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DLXXX PRACE MATEMATYCZNE, ZESZYT 22 1981

A Modified Edge of the Wedge Theorem

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Abstract. A modified version of the edge of the wedge theorem due to O. Stormark is improved by showing that some conditions of his theorem are superfluous.

In the Stormark paper [2] the boundary values are required to be equal along a bunch of complex lines L (see Condition B below) and to be analytically dependent on L; see Condition C of [2]. We show that the Condition C can be dropped. Moreover the Condition D of [2] can be replaced by a weaker one (see Condition D′ below). Namely we prove the following

THEOREM. *Let f be a holomorphic function in the open set $G := Q \times i\Gamma \subset C^{1+n}$, where*

$$Q := \{x \in R^{1+n}: |x_j| < 1,\ j = 0, \dots, n\},$$
$$\Gamma := \{y \in R^{1+n}: 0 < |y_0| < 1,\ |y_k| < |y_0|\ (k = 1, \dots, n)\}.$$

Let f satisfy the following two conditions:

CONDITION B. *For every $u \in R^n$ with $|u_j| \leqslant \varepsilon$ ($j = 1, \dots, n$), where ε is fixed and $0 < \varepsilon < 1$, the function*

$$g_u : z_0 \mapsto f(z_0, u_1 z_0, \dots, u_n z_0)$$

that is holomorphic in the set $\{z_0 \in C: |z_0| < 1,\ y_0 = Im\, z_0 \neq 0\}$ can be continued to a holomorphic function in the unit disc $\{|z_0| < 1\}$.

CONDITION D′. *For every multiindex $\mu \in Z^n_+$ one can find constants $M_\mu > 0$ and $0 < r_\mu < 1$ such that*

$$\left|\frac{\partial^\mu f}{\partial z^\mu}(z_0, 0)\right| \leqslant M_\mu \quad \text{when } |z_0| < r_\mu \text{ and } y_0 \neq 0, \tag{1}$$

where $z = (z_1, \dots, z_n)$.

Then there exists a unique holomorphic function F in a neighborhood P of $0 \in C^{1+n}$ such that $F|_{P \cap G} = f|_{P \cap G}$.

Proof. 1° By condition B the function

$$g(w_0, \dots, w_n) := f(w_0, w_0 w_1, \dots, w_0 w_n)$$

is well defined and separately holomorphic on the set

$$X := (D_0 \times E_1 \times \dots \times E_n) \cup (E_0 \times D_1 \times E_2 \times \dots \times E_n) \cup \dots \cup (E_0 \times \dots \times E_{n-1} \times D_n),$$

where

$$D_0 = \{w_0 = u_0 + iv_0 \in C\colon\ 0<|w_0|<1\}\,,\qquad E_0 = \{iv_0\colon\ v_0 \in R,\ \tfrac{1}{4}\leqslant |v_0| \leqslant \tfrac{3}{4}\}\,,$$
$$D_j = \{w_j = u_j + iv_j \in C\colon\ |w_j|<1\}\,,\qquad E_j = \{u_j \in R\colon\ |u_j| \leqslant \varepsilon\}\,.$$

Here the separate holomorphicity means that for every j the function g is holomorphic with respect to $w_j \in D_j$ whilst the remaining variables are fixed in the sets E_k $(k \neq j)$.

Let h_j be the solution of the Dirichlet problem for the domain $D_j \setminus E_j$ with the boundary values 0 on E_j and 1 on ∂D_j. Then by theorem 7.1 of [1] there exists a function $\tilde{g}$ holomorphic in the domain

$$\Omega := \{w \in D_0 \times \ldots \times D_n\colon\ h_0(w_0) + \ldots + h_n(w_n) < 1\}$$

such that $\tilde{g}|_{\Omega \cap X} = g|_{\Omega \cap X}$. Now take $d>0$ so small that the domain

$$V := \{w \in C^{1+n}\colon\ 0<|w_0|<d,\ |w_j|<d\ (j = 1, \ldots, n)\}$$

is contained in Ω. It is now obvious that the function

$$\tilde{f}(z_0, \ldots, z_n) := g\left(z_0, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)$$

is holomorphic in the complex cone

$$K := \{(z_0, z) \in C^{1+n}\colon\ 0<|z_0|<d,\ |z_j|<d|z_0|\ (j = 1, \ldots, n)\}$$

and moreover

(2) $$\tilde{f}|_{K \cap G} = f|_{K \cap G}\,.$$

2° So far we have used only Condition B. Now, using Condition D′, we shall get the holomorphic continuation F of $\tilde{f}$ (and consequently the holomorphic continuation of f) to the polydisc

$$P := \{(z_0, z) \in C^{1+n}\colon\ |z_0|<d,\ |z_j|<d^2\ (j = 1, \ldots, n)\}\,.$$

Indeed, the function $\tilde{f}$ can be represented in the form

(3) $$\tilde{f}(z_0, z) = \sum_{\mu \in Z_+^n} f_\mu(z_0) z^\mu,\qquad (z_0, z) \in K\,,$$

where the coefficient

(4) $$f_\mu(z_0) = \frac{1}{\mu!}\frac{\partial^\mu \tilde{f}}{\partial z^\mu}(z_0, 0) = \frac{1}{\mu!}\frac{\partial^\mu f}{\partial z^\mu}(z_0, 0)$$

is holomorphic in the set $\{0<|z_0|<1\}$. By (1), (2) and (4) the function f_μ is continuable to a function $\tilde{f}_\mu$ holomorphic in the unit disc $\{|z_0|<1\}$. The multiple Hartogs series (3) is uniformly convergent on every compact subset of K. Therefore the desired continuation F is given by

$$F(z_0, z) := \sum_\mu \tilde{f}_\mu(z_0) z^\mu,\qquad |z_0|<d,\ |z_j|<d^2\quad (j = 1, \ldots, n)\,.$$

COROLLARY. *A function f holomorphic in the complex cone K is continuable to a holomorphic function in the polydisc P if and only if f satisfies Condition* D'.

Remarks. 1° In the definition of the "cross" X we could take the set E_0 of the form

$$E_0 = \{iv_0 \colon v_0 \in \mathbf{R},\ \delta \leqslant |v_0| \leqslant 1-\delta\},$$

where δ is any real number with $0<\delta<\frac{1}{2}$.

2° By a suitable change of coordinates one can easily check that our theorem remains true if $G = U \times i\Gamma$, where $\Gamma = (\Gamma_+ \cup \Gamma_-) \cap B$, $\Gamma_- = -\Gamma_+$, $\Gamma_- \cap \Gamma_+ = \emptyset$, Γ_+ is an open connected cone and B is a ball in $\mathbf{R}^{1+n}$ with the center at the origin.

References

[1] J. Siciak, *Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of C^n*, Ann. Polon. Math. 22 (1969), 145–171.

[2] O. Stormark, *A local edge of the wedge theorem*, Commun. Math. Phys. 43 (1975), 33–37.

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