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Continuity of solutions of elliptic equations of the high order with parameter

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Abstract. We shall consider a differential operator $L = \sum_{|\alpha| \leq r} a_\alpha D^\alpha$, $r = 2m$, of the high order with real coefficients a_α of class $C^{|\alpha|+r+n+1}$ in the closure $\bar{\Omega}$ of a bounded domain Ω with C^∞ boundary $\partial\Omega$. We shall consider the continuous dependence of solutions of the Dirichlet problem for the equation $Lu - \lambda u = f_\lambda$ with respect to both x and parameter λ.

Let Ω be a bounded domain in R^n with C^∞ boundary $\partial\Omega$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a sequence of n non-negative integers and $\xi = (\xi_1, \ldots, \xi_n)$ is any n dimensional vector, we set

$$\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n .$$

Similarly

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \quad \text{where } D_j = \frac{\partial}{\partial x_j}, \; j = 1, \ldots, n .$$

We denote by $H^k(\Omega)$ the k-th Sobolev's space, i.e. the set of distributions in Ω such that all derivatives of order $\leq k$ of u belong to $L^2(\Omega)$. The space $H^k(\Omega)$ is equipped with the standard norm $\|\cdot\|_k$ associated with the sesquilinear form

$$(u, v)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \, D^\beta v \, dx .$$

Note that C_0^∞ is not dense in $H^k(\Omega)$, for $k > 0$ (this is only true in the case of $k = 0$, because $H^0(\Omega) = L^2(\Omega)$). We shall denote by $H_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$. As is well known, $H_0^k(\Omega)$ is a proper Hilbert subspace of $H^k(\Omega)$.

In the forthcoming we shall consider a partial differential operator

$$L = \sum_{|\alpha| \leq r} a_\alpha D^\alpha$$

of high order $r = 2m > \frac{n}{2}$ with real coefficients and with domain $\mathscr{D}_m = H_0^m \cap H^{2m}$.

We shall assume that

1° L is a strongly elliptic operator, i.e. there is a constant $K>0$ such that

$$\sum_{|\alpha|=r} a_\alpha \xi^\alpha \geqslant K|\xi|^r .$$

2° coefficients a_α belong to $C^{|\alpha|+r+n+1}(\overline{\Omega})$.

3° L is symmetric, i.e. for every $u, v \in \mathscr{D}_m$

$$(Lu, v)_0 = (u, Lv)_0$$

4° L is positively defined, i.e. there is a constant $\gamma>0$ such that

$$(Lu, u)_0 \geqslant \gamma(u, u)_0, \quad \text{for } u \in \mathscr{D}_m .$$

Let I be the identity map on $\mathscr{D}_m$ and $A_\lambda = L-\lambda I$, $\lambda \in \boldsymbol{R}$. We shall consider the Dirichlet problem for equation

$$A_\lambda u = f_\lambda, \quad \lambda \in R \tag{1}$$

with right-hand side $f_\lambda \in L^2(\Omega)$. This means to find $u_\lambda \in \mathscr{D}_m$ such that $A_\lambda u_\lambda = f_\lambda$, $\lambda \in \boldsymbol{R}$.

With notations as before we have the following

THEOREM 1. *Let $\lambda_1, \lambda_2, \ldots$ be the sequence of eigenvalues of the operator L and let the map*

$$\boldsymbol{R}-\{\lambda_j\} \ni \lambda \to f_\lambda \in L^2(\Omega)$$

be continuous. If $u_\lambda \in \mathscr{D}_m$ is a solutions of the Dirichlet problem for equation (1) *then the map*

$$(\boldsymbol{R}-\{\lambda_j\}) \times \Omega \ni (\lambda, x) \to u_\lambda(x) \in \boldsymbol{R}$$

is continuous.

In order to prove Theorem 1 we shall use the following known facts

1) The sequence $\{\lambda_j\}$ of eigenvalues of L is a sequence of positive numbers, increasing and converging to infinity.

2) The equation $A_\lambda u = 0$ has a non-zero solution in $\mathscr{D}_m$ if and only if there is $j \in \boldsymbol{N}$ such that $\lambda = \lambda_j$. This non-zero solution is called the eigenfunction of L corresponding to the eigenvalue λ_j.

3) Operator

$$A_{\lambda_j}\colon \mathscr{D}_m \to L^2(\Omega)$$

is linear with a non trivial kernel V_j. The kernel V_j is called the eigenspace corresponding to the eigenvalue λ_j.

4) The subspaces $V_1, V_2, \ldots$ are finite dimensional and V_j are pairwise orthogonal in $L^2(\Omega)$.

5) It follows from 4) that we would be able to find an orthonormal sequence of eigenfunctions $u_1, u_2, \ldots$ such that

$u_1, \ldots, u_{k_1}$ is an orthonormal basis in V_1,
$u_{k_1+1}, \ldots, u_{k_2}$ is an orthonormal basis in V_2,
. . . .

6) The sequence $u_1, u_2, \dots$ is complete in $L^2(\Omega)$.

7) Eigenfunctions belong to $C^{2m}(\Omega)$.

8) If $f_\lambda \in L^2(\Omega)$ then there is $u \in \mathscr{D}_m$ such that $A_\lambda u = f_\lambda$. If $f_\lambda \in H^{[n/2]+1}(\Omega)$ then $u \in H^{2m+[n/2]+1}(\Omega) \subset C^{2m}(\Omega)$.

If $\lambda \in \boldsymbol{R} - \{\lambda_j\}$ then it follows from 2) and 8) that the map

$$\mathscr{D}_m \ni u \rightarrow A_\lambda u \in L^2(\Omega)$$

is bijective and therefore A_λ is invertible. The operator $G_\lambda = A_\lambda^{-1}$ is said to be the Green's operator for A_λ. Now, we recall the known theorems about G_λ.

9) G_λ is a bounded operator and there exists a function R_λ defined in $(\Omega \times \Omega) - \{(x, y): x = y\}$ such that

$$(G_\lambda u, v)_0 = \iint_{\Omega \times \Omega} R_\lambda(x, y) u(x) v(y) \, dx \, dy$$

for all continuous functions u, v with compact supports in Ω. The function R_λ is called the Green's function for A_λ.

10) Inasmuch as operator A_λ is the high order and satisfies 1°–4° we have

a) $R_\lambda(x, \cdot) \in L^2(\Omega)$ for all $x \in \Omega$,

b) $R_\lambda(\cdot, y) \in L^2(\Omega)$ for all $y \in \Omega$,

c) Integrals

$$\int_\Omega |R_\lambda(x, y)|^2 dy \quad \text{and} \quad \int_\Omega |R_\lambda(x, y)|^2 dx$$

are locally bounded in Ω.

11) If $\lambda \in \boldsymbol{R} - \{\lambda_j\}$ and f_λ is a continuous function belongs to $\mathscr{D}_m$ then the solution u_λ of the Dirichlet problem for equation $A_\lambda u = f_\lambda$ has the form

$$u_\lambda(x) = \int_\Omega R_\lambda(x, y) f_\lambda(y) \, dy .$$

Proof of Theorem 1. Inasmuch as eigenfunctions $u_1, u_2, \dots$ forms an orthonormal and complete system in $L^2(\Omega)$ and the map $\lambda \rightarrow f_\lambda$ is continuous we can present f_λ in the form

$$f_\lambda = \sum_{j=1}^{\infty} c_j(\lambda) u_j ,$$

where coefficients $c_j(\lambda) = (f_\lambda, u_j)$, $j = 1, \dots$ are continuous in $\boldsymbol{R} - \{\lambda_j\}$, because the map $\lambda \rightarrow f_\lambda$ is continuous.

It follows from 8) that u_λ is unique. Observe that

$$\text{(i)} \qquad u_\lambda = \sum_{j=1}^{\infty} \frac{c_j(\lambda)}{\lambda - \lambda_j} u_j$$

as an element in $L^2(\Omega)$.

Let $\lambda_0 \in \boldsymbol{R} - \{\lambda_j\}$ and let $\delta > 0$ be so chosen that $I = [\lambda_0 - \delta, \lambda_0 + \delta] \subset \boldsymbol{R} - \{\lambda_j\}$. We shall show that for every compact $K \subset \Omega$ the series (i) is uniformly convergent in $I \times K$, by showing the Cauchy condition.

Let us take $(\lambda, x) \in I \times K$ and let

$$B_{mn}(\lambda, x) = \left|\sum_{j=n}^{n+m} \frac{c_j(\lambda)}{\lambda - \lambda_j} u_j(x)\right|.$$

It follows from the Schwarz inequality that

$$B_{mn}^2(\lambda, x) \leqslant \left(\sum_{j=n}^{n+m} \left|\frac{u_j(x)}{\lambda_j}\right|^2\right) \left(\sum_{j=n}^{n+m} \left|\frac{c_j(\lambda)}{\frac{\lambda}{\lambda_j} - 1}\right|^2\right).$$

Applying 11) we have

$$\frac{1}{\lambda_j} u_j(x) = \int_{\Omega} R_0(x, y) u_j(y) dy.$$

Hence $\frac{1}{\lambda_j} u_j(x)$ is the j-th Fourier coefficient for the function $R_0(x, \cdot)$ and by the Bessel's inequality

$$\sum_{j=1}^{\infty} \frac{u_j^2(x)}{\lambda_j^2} \leqslant \|G(x, \cdot)\|_0^2 \leqslant M,$$

for $x \in K$ and for a constant $M > 0$, because of c) in 10). Moreover, there is a constant $B > 0$ such that for all $\lambda \in I$ and $j = 1, 2, \ldots$ inequality $\left|\frac{\lambda}{\lambda_j} - 1\right| > B$ holds. Thus we have

$$B_{mn}^2(\lambda, x) \leqslant MB \sum_{j=n}^{n+m} c_j^2(\lambda), \qquad \lambda \in I,\ x \in K.$$

On the other hand

$$\sum_{j=1}^{\infty} c_j^2(\lambda) = \|f_\lambda\|_0^2$$

and the proof of Cauchy's condition of uniform convergent is complete if we apply Dini's theorem to the series $\sum^{\infty} c_j^2(\lambda)$ of functions c_j^2 continuous in I.

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