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Derivation of the diffusion equation in the space of distributions

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We shall derive in this note the one-dimensional diffusion equation in an infinite homogenous cylinder with a constant cross section S, in the case when a solution of that equation is not differentiable at $x = 0$. Let the axis of the cylinder be the x-axis. Let us introduce a coordinate system x, t (t is time variable). Let $p(x, t)$ be the density. Let $p(x, 0)$ be non-negative in the interval $[-\lambda, \lambda]$, symmetrical with respect to ist centre, i.e. $p(x, 0) = p(-x, 0)$ for all x. Assume moreover that $\lim\limits_{x\to 0^+} \frac{\partial p(x, 0)}{\partial x}$ is finite, and $p(x, 0) = 0$ for $|x| > \lambda$. Since the function $p(x, t)$ is not differentiable at $x = 0$, it cannot be a solution of the diffusion equation in the classical sense. We will associate with it a distribution considered as a function of x and will find a differential equation describing transport in the space of distributions T: $D \to R$. D is the space of infinitely differentiable functions equal to zero outside a compact set (depending on the function). Let us define the distribution T_p associated with the function p by

$$\varphi \in D \to T_p(\varphi) = \langle p, \varphi \rangle = \int_{-\infty}^{\infty} p(x, t)\varphi(x)dx\,.$$

The integral is taken in the sense of Lebesgue.

Assume that

(1) $p(x, t)$ is continuous for $0 \leqslant t < \infty$, $-\infty < x < \infty$.

(2) $\frac{\partial p(x, t)}{\partial x}$, $\frac{\partial^2 px, t}{\partial x^2}$ are bounded with respect to x for every t.

(3) $p(x, t)$ is of class C^1 with respect to t and class C^2 with respect to x, $x \neq 0$.

In order to derive the equation let us consider an element of the cylinder contained between cross sections placed at $x-h$ and $x+h$. Let us formulate the balance of the flux. Inflow to the element at the point $x-h$ in time Δt lessened by the outflow at the point $x+h$ in the same time is equal to the change in the contents of the element. We are going now to determine those three parts of the balance. Inflow in the time Δt equals to the flux (the amount flowing in per unit of time) multiplied by the time Δt. We can show that there exists a coefficient $c(x, t)$ such that the flux is equal to [1]

$$-kS\left(\frac{\partial p}{\partial x} + c(x, t)p(x, t)\right), \quad k = \text{const} > 0\,.$$

Let us assume that

(4) $c(x, t)$ is of class C^1 with respect to x, $x \neq 0$, its derivative $\frac{\partial c}{\partial x}$ being bounded,

(5) $c(x, t) = -c(-x, t)$.

Thus the inflow equals to

$$-kS\left[\frac{\partial p(x-h, t)}{\partial x}+c(x-h, t)p(x-h, t)\right]\cdot \Delta t .$$

Similarly, the outflow at the point $x+h$ is equal to

$$-kS\left[\frac{\partial p(x+h, t)}{\partial x}+c(x+h, t)p(x+h, t)\right]\cdot \Delta t .$$

Now, let us calculate the change of the contents of the element. ΔQ is proportional to the product of the volume of the element and of the change Δp in the time Δt. Therefore, it can be written as $\Delta Q = \bar{c}2hS\frac{\partial p}{\partial t}\Delta t$. In calorimetry $\bar{c} = c_w \cdot \varrho$, where c_w is the specific heat and ϱ is the density of cylinder.

The distributions corresponding to the inflow and outflow are

$$-kS \int_{-\infty}^{\infty}\left[\frac{\partial p(x-h, t)}{\partial x}+c(x-h, t)p(x-h, t)\right]\varphi(x)dx\Delta t ,$$

$$-kS \int_{-\infty}^{\infty}\left[\frac{\partial p(x+h, t)}{\partial x}+c(x+h, t)p(x+h, t)\right]\varphi(x)dx\Delta t ,$$

while the distribution corresponding to the change in the contents of the element is

$$\bar{c}\cdot S\cdot 2h\cdot \Delta t \int_{-\infty}^{\infty}\frac{\partial p(x, t)}{\partial t}\varphi(x)dx .$$

Since inflow $-$ outflow $=$ change in the contents of the element [2], the distributions associated with these quantities are equal. It can be written

$$\text{(6)}\quad \bar{c}\cdot 2h\cdot \Delta t \int_{-\infty}^{\infty}\frac{\partial p(x, t)}{\partial t}\varphi(x)dx = k\cdot \Delta t\left\{-\int_{-\infty}^{\infty}\left[\frac{\partial p(x-h, t)}{\partial x}+\right.\right.$$

$$\left.\left.+c(x-h, t)p(x-h, t)\right]\varphi(x)dx+\int_{-\infty}^{\infty}\left[\frac{\partial p(x+h, t)}{\partial x}+c(x+h, t)p(x+h, t)\right]\varphi(x)dx\right\}.$$

Let $k = \bar{c}$. Equation (6) can be written as follows

$$(7)\qquad -\int_{-\infty}^{\infty}\left[\frac{\partial p(x-h,t)}{\partial x}+c(x-h,t)p(x-h,t)\right]\varphi(x)dx+\int_{-\infty}^{\infty}\left[\frac{\partial p(x+h,t)}{\partial x}+\right.$$
$$\left.+c(x+h,t)p(x+h,t)\right]\varphi(x)dx = 2h\int_{-\infty}^{\infty}\frac{\partial p(x,t)}{\partial t}\varphi(x)dx\,.$$

Notice that the left-hand side of (7) has the form

$$(8)\qquad \int_{-\infty}^{\infty}[g(x+h,t)-g(x-h,t)]\varphi(x)dx \quad \text{with } g(x,t) = \frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)\,.$$

We can write

$$\int_{-\infty}^{\infty}[g(x+h,t)-g(x-h,t)]\varphi(x)dx = \int_{-\infty}^{-h}[g(x+h,t)-g(x-h,t)]\varphi(x)dx+$$
$$+\int_{h}^{\infty}[g(x+h,t)-g(x-h,t)]\varphi(x)dx+\int_{-h}^{h}[g(x+h,t)-g(x-h,t)]\varphi(x)dx\,.$$

By the assumptions of the theorem the derivative of the function g with respect to x is bounded whence it follows that g satisfies a Lipschitz condition. Consequently we have

$$(9)\qquad \lim_{h\to 0}\frac{1}{h}\int_{-h}^{h}[g(x+h,t)-g(x-h,t)]\varphi(x)dx = 0\,.$$

On the other hand, applying the mean value theorem we get

$$(10)\qquad \int_{-\infty}^{-h}[g(x+h,t)-g(x-h,t)]\varphi(x)dx+\int_{h}^{\infty}[g(x+h,t)-g(x-h,t)]\varphi(x)dx$$
$$= 2h\int_{-\infty}^{-h}g_x(x+\theta_1(x)h,t)\varphi(x)dx+2h\int_{h}^{\infty}g_x(x+\theta_2(x)h,t)\varphi(x)dx\,.$$

Relations (9) and (10) imply that dividing (7) by $2h$ and letting h tend to 0 we get by Lebesgue's theorem in the limit

$$\int_{-\infty}^{\infty}\frac{\partial p(x,t)}{\partial t}\varphi(x)dx = \int_{-\infty}^{\infty}\frac{\partial g(x,t)}{\partial x}\varphi(x)dx$$

or explicity

$$(11)\qquad \int_{-\infty}^{\infty}\frac{\partial p(x,t)}{\partial t}\varphi(x)dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)\right]\varphi(x)dx\,.$$

The left hand side of (11) is equal to

$$\frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty} p(x,t)\varphi(x)dx.$$

The last expression is the time derivative of the distribution associated with the function p (as function of x) and can be shortly denoted $\frac{\partial p}{\partial t}$. The right hand side of (11) is the distribution associated with the derivative $\frac{\partial}{\partial x}$ of the function $\frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)$. The last function has a jump at the point $x = 0$, which is equal to

$$-2\left[\frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)\right]_{x=0^-}$$

Consequently, by the well-known theorem the right hand side of (11) can be written as [3]

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial}{\partial x}(c\cdot p)+2\left[\frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)\right]_{x=0^-}\cdot\delta_0,$$

where the derivatives are meant in the distributional sense and δ_0 is the function of Dirac. Thus we get finally from (11)

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p}{\partial x^2}+\frac{\partial}{\partial x}(c\cdot p)+2\left[\frac{\partial p(x,t)}{\partial x}+c(x,t)p(x,t)\right]_{x=0^-}\cdot\delta_0. \tag{12}$$

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References

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