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On a certain property of the boundary for the diffusion equation

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It will be shown that for the partial differential equation

(1) $$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}(c \cdot p) + 2\left[\frac{\partial p}{\partial x} + c(x, t)p(x, t)\right]_{x=0^-} \cdot \delta_0,$$

where δ_0 is the Dirac's distribution, there exists a coefficient $c(x, t)$ continuous in the strip

(2) $$0<t<r, \quad -\infty<x<\infty$$

and a continuous function $p(x, t)$ which satisfies the equation (1) in the strip (2) in distribution sense and which has the following properties:

1° the boundary on which

(3) $$p(x, t) = 0$$

is given by the equation

(4) $$|x| = \lambda(t), \quad 0<t<r,$$

where

(5) $$\lambda(t)>0, \quad \frac{d\lambda(t)}{dt}<0$$

2° the following condition

(6) $$\int_{-\lambda(t)}^{\lambda(t)} p(x, t)dx = \text{const}$$

is satisfies. Put $c(x, t) = \dfrac{x}{2(r-t)}$ and

(7) $$p(x, t) = -2\frac{|x|}{r-t} + 2(r-t)^{-1/2}.$$

From the equation $p(x, t) = 0$ we get $|x| = \sqrt{r-t}$, hence $\lambda(t) = \sqrt{r-t}$. It is clear that the conditions (5) are satisfied. The condition (6) holds as well, since

$$\int_{-\sqrt{r-t}}^{0}\left[\frac{2x}{r-t}+2(r-t)^{-1/2}\right]dx = 1\,.$$

For $x \neq 0$ the function (7) satisfies the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}+\frac{\partial}{\partial x}\,[c(x,t)p(x,t)]$$

in the classical sense because

$$\frac{\partial p}{\partial t}-\frac{\partial^2 p}{\partial x^2} = 2\frac{|x|}{(r-t)^2}+(r-t)^{-3/2}$$

and

$$\frac{\partial}{\partial x}\,[c(x,t)p(x,t)] = \frac{\partial}{\partial x}\left[\frac{x}{2(r-t)}\left(-2\frac{|x|}{r-t}+2(r-t)^{-1/2}\right)\right] = 2\frac{|x|}{(r-t)^2}+(r-t)^{-3/2}\,.$$

Check that the function c is continuous, $c(0,t) = 0$ and the derivatives $\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x}, \frac{\partial^2 p}{\partial x^2}$ are locally integrable in the region (2), where the function p is continuous whereas $\frac{\partial p}{\partial x}$ has a jump at $x = 0$. Therefore, the distribution associated with $p(x,t)$ satisfies the equation (1), in which the derivatives are considered as distribution derivatives.

We can seek a solution of the equation (1) in the form of a series

$$p(x,t) = \frac{1}{2\sqrt{r-t}}\sum_{j=0}^{\infty} a_j\left(1-\frac{x^2}{r-t}\right)^{j+1}.$$

Substituting into (1) we obtain the difference equation

$$a_{j+2}-a_{j+1}\frac{2j^2+7j+6}{2j^2+10j+12} = 0$$

and from it

$$a_{j+1} = \frac{\Gamma(j+\frac{3}{2})}{\Gamma(j+3)}.$$

The convergence radius of the series $\sum_{j=0}^{\infty} a_j z^{j+1}$ is 1.

Computing the sum of the series we obtain (7).

A solution to the problem of diffusion satisfying the conditions (3)–(6) has not been published yet.

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