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On approximation and interpolation of entire functions

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1. Introduction. Let K be a compact subset of the complex plane C, of positive logarithmic capacity. Let f be a complex function defined and bounded on K. For $n \in N$ put

$$E_n(f, K) = \|f - t_n\|,$$

where the norm is the maximum norm on K and t_n denotes the n-th Čebyšev polynomial of the best approximation to f on K.

It is known (see [4]) that

$$\lim_{n\to 0} (E_n(f, K))^{1/n} = 0$$

if and only if f is the restriction to K of an entire function g. The aim of this paper is to establish relations between the rate at which $(E_n(f, K))^{1/n}$ tends to zero and the growth of g.

Let g be an entire transcendental function. Following A. Schönhage [3] we examine the growth of the function g with the aid of quantities $\varrho(k, g)$, $\sigma(k, g)$, $\varrho_l(g)$ and $\sigma_l(g)$ defined below.

Put

$$M(r, g) = \max\{|g(z)|: |z| = r\}, \quad r>0.$$

Let k be a positive integer. Write

$$\ln_k x = \ln(\ln_{k-1} x), \quad \ln_0 x = x.$$

Observe that $\ln_k x>0$ for all sufficiently large positive x.

Definition 1.1.

$$\varrho(k, g) = \limsup_{r\to\infty} (\ln r)^{-1} \ln_{k+1} M(r, g).$$

We have $0\leqslant\varrho(k, g)\leqslant+\infty$. If $0<\varrho(k, g)<+\infty$ then we define

$$\sigma(k, g) = \limsup_{r\to\infty} r^{-\varrho(k,g)} \ln_k M(r, g).$$

If $k = 1$, we obtain the classical definitions of order and type.

Definition 1.2. We say that g is of *index* Ig if

$$Ig = \inf\{k \in N:\ \varrho(k, g) < +\infty\}.$$

Remark. If there exists a positive integer k such that $\varrho(k, q) \in (0, +\infty)$, then $Ig = k$.

Definition 1.3.

$$\varrho_l(g) = \limsup_{r\to\infty} (\ln\ln r)^{-1} \ln\ln M(r, g).$$

We have $1 \leqslant \varrho_l(g) \leqslant +\infty$. If $\varrho_l(g)$ is finite, then we define

$$\sigma_l(g) = \limsup_{r\to\infty} (\ln r)^{-\varrho_l(g)} \ln M(r, g).$$

Now we may state the main result.

THEOREM 1.1. *Let f be a function defined and bounded on a compact set K of positive logarithmic capacity d and let k be a positive integer. Put $E_n = E_n(f, K)$. Then*

(i)
$$\alpha = \limsup_{n\to\infty} \frac{\ln_k n}{\ln\left(\dfrac{d}{E_n^{1/n}}\right)}$$

satisfies $0 < \alpha < +\infty$ *if and only if f is the restriction to K of an entire function g with* $\varrho(k, g) = \alpha$.

(ii) For $\alpha \in (0, +\infty)$ *we have*:

$$\beta = \limsup_{n\to\infty} (\ln_{k-1} n)^{1/\alpha} E_n^{1/n}$$

satisfies $0 < \beta < +\infty$ *if and only if f is the restriction to K of an entire function g of index k, with* $\varrho(k, g) = \alpha$ *and*

$$\sigma(k, g) = \begin{cases} \dfrac{1}{e\alpha}\left(\dfrac{\beta}{d}\right)^{\alpha}, & k = 1, \\[2ex] \left(\dfrac{\beta}{d}\right)^{\alpha}, & k \geqslant 2. \end{cases}$$

(iii)
$$\alpha = \limsup_{n\to\infty} \frac{\ln n}{\ln\ln\left(\dfrac{d}{E_n^{1/n}}\right)}$$

satisfies $0 < \alpha < +\infty$ *if and only if f is the restriction to K of an entire function g with* $\varrho_l(g) = \alpha + 1$.

(iv) For $\alpha \in (0, +\infty)$ *we have*:

$$\beta = \limsup_{n\to\infty} \frac{n^{1/\alpha}}{\ln\left(\dfrac{d}{E_n^{1/n}}\right)}$$

satisfies $0<\beta<+\infty$ *if and only if* f *is the restriction to* K *of an entire function* g *with* $\varrho_l(g)=\alpha+1$ *and*

$$\sigma_l(g)=\frac{(\alpha\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}}.$$

We also state that in Theorem 1.1 $E_n(f, K)$ may be replaced by $\|f-l_n\|$ or $\|l_n-l_{n-1}\|$, l_n being the n-th Lagrange interpolation polynomial for f with nodes at extremal points of K (see [1]).

In the case of $k=1$ T. Winiarski [5] obtained (ii) and partly (i) of Theorem 1.1 for $E_n(f, K)$ as well as for $\|f-l_n\|$ and $\|l_n-l_{n-1}\|$. If $K=[-1,1]$, Theorem 1.1 was proved by A. R. Reddy [2].

2. Estimates of $\sigma(k, g)$, $\varrho(k, g)$, $\varrho_l(g)$ and $\sigma_l(g)$. Henceforth K is a compact set with positive capacity d and $\|\cdot\|$ denotes the maximum norm on K.

Let Φ be the extremal function of K (see [1]). For $r>d$ put

$$K_r=\{z\in C\colon d\Phi(z)<r\},$$
$$C_r=\{z\in C\colon d\Phi(z)=r\}.$$

Let g be an entire function. Write

$$\tilde{M}(r, g)=\max\{|g(z)|\colon z\in C_r\},\qquad r>d.$$

LEMMA 2.1. *If* g *is a transcendental entire function, then in Definitions* 1.1 *and* 1.3 *we may replace* $M(r, g)$ *by* $\tilde{M}(r, g)$.

Proof — proceeds on the same lines as that of Lemma 3.1 in [5].

Let $n\in N$ and let P_n denote the set of all complex polynomials in C of degree not exceeding n.

Fix a sequence of polynomials $(p_n)_{n\in N}$, where $p_n\in P_n$ for every n, such that for each $r>0$ the set $\left\{\|p_n\|\left(\frac{r}{d}\right)^n\colon n\in N\right\}$ is bounded. Note that the function $\sum_{n=1}^{\infty}p_n$ is then entire.

Put

$$M^*(r)=M^*(r, (p_n)_{n\in N})=\sup\left\{\|p_n\|\left(\frac{r}{d}\right)^n\colon n\in N\right\},\qquad r>0.$$

Analogously to Definition 1.1 put for $k\in N$

DEFINITION 2.1.

$$\varrho^*(k)=\varrho^*(k, (p_n)_{n\in N})=\limsup_{r\to\infty}(\ln r)^{-1}\ln_{k+1}M^*(r).$$

If $\varrho(k)=\varrho(k, \sum_{n=1}^{\infty}p_n)$ is positive and finite, we define

$$\sigma^*(k)=\sigma^*(k, (p_n)_{n\in N})=\limsup_{r\to\infty}r^{-\varrho(k)}\ln_k M^*(r).$$

Analogously to Definition 1.3 put

Definition 2.2.

$$\varrho_l^* = \varrho_l^*((p_n)_{n\in N}) = \limsup_{r\to\infty} (\ln\ln r)^{-1} \ln\ln M^*(r).$$

If $\varrho_l = \varrho_l(\sum_{n=1}^{\infty} p_n)$ is finite, we define

$$\sigma_l^* = \sigma_l^*((p_n)_{n\in N}) = \limsup_{r\to\infty} (\ln r)^{-\varrho_l} \ln M^*(r).$$

LEMMA 2.2.
(i) $\varrho(k) \leqslant \varrho^*(k)$;
(ii) $\sigma(k) \leqslant \sigma^*(k)$ *provided* $\varrho(k) \in (0, +\infty)$;
(iii) $\varrho_l \leqslant \varrho_l^*$;
(iv) $\sigma_l \leqslant \sigma_l^*$ *provided* $\varrho_l \in [1, +\infty)$.

(Here k is a fixed positive integer, $\sigma(k)$ *and* σ_l *stand for* $\sigma(k, \sum_{n=1}^{\infty} p_n)$ *and* $\sigma_l(\sum_{n=1}^{\infty} p_n)$, *respectively).*

Proof. Observe that for every $\delta > 1$ there exists a real number $\theta(\delta)$ such that for $r > d$

$$\tilde{M}(r) \leqslant \theta(\delta) M^*(\delta r) \tag{2.1}$$

(here $\tilde{M}(r)$ stands for $\tilde{M}(r, \sum_{n=1}^{\infty} p_n)$).

Indeed, by the Bernstein–Walsh inequality [4]

$$|p_n(z)| \leqslant \|p_n\| \left(\frac{r}{d}\right)^n \quad \text{for } z \in C_r,\ n \in N.$$

Then

$$\tilde{M}(r) \leqslant \sum_{n=1}^{\infty} \|p_n\| \left(\frac{r}{d}\right)^n.$$

Writing $\left(\frac{r}{d}\right)^n = \left(\frac{\delta r}{d}\right)^n \delta^{-n}$ we obtain

$$\tilde{M}(r) \leqslant \sum_{n=1}^{\infty} \|p_n\| \left(\frac{\delta r}{d}\right)^n \delta^{-n}.$$

Therefore by definition of $M^*(r)$

$$\tilde{M}(r) \leqslant \sum_{n=1}^{\infty} M^*(\delta r) \delta^{-n}.$$

So we get (2.1) with $\theta(\delta) = \sum_{n=1}^{\infty} \delta^{-n} < +\infty$.

From (2.1) we obtain (with $\delta = 2$)

$$\frac{\ln_{k+1}\tilde{M}(r)}{\ln r} \leqslant \frac{\ln_{k+1} M^*(2r)}{\ln(2r)} \cdot \frac{\ln(2r)}{\ln r}.$$

After passing to the upper limit we get

$$\varrho(k) \leqslant \varrho^*(k).$$

In order to prove (ii) fix $\varepsilon > 0$ and $\delta > 1$. Again from (2.1) we obtain

$$r^{-\varrho(k)}\ln_k \tilde{M}(r) \leqslant r^{-\varrho(k)}\ln_k[\theta(\delta)M^*(\delta r)], \quad r > d.$$

Hence there exists an $R(\varepsilon, \delta)$ such that for $r > R(\varepsilon, \delta)$

$$r^{-\varrho(k)}\ln_k \tilde{M}(r) \leqslant r^{-\varrho(k)}(1+\varepsilon)\ln_k M^*(\delta r).$$

This inequality can be rewritten in the form

$$r^{-\varrho(k)}\ln_k \tilde{M}(r) \leqslant (1+\varepsilon)\delta^{\varrho(k)}(\delta r)^{-\varrho(k)}\ln_k M^*(\delta r).$$

After passing to the upper limit we get

$$\sigma(k) \leqslant (1+\varepsilon)\delta^{\varrho(k)}\sigma^*(k)$$

and because of arbitrariness of ε and δ

$$\sigma(k) \leqslant \sigma^*(k).$$

The proofs of (iii) and (iv) proceed on the same lines as those of (i) and (ii) respectively.

LEMMA 2.3. *Let $(q_n)_{n\in N}$ be a sequence of polynomials of the respective degrees not exceeding n, and let k be a positive integer. If there exist positive numbers α, β and n_0 such that for every $n > n_0$*

$$\|q_n\| \leqslant d^n \left(\frac{\beta}{\ln_{k-1} n}\right)^{n/\alpha},$$

then

(i) *$\sum_{n=1}^{\infty} q_n$ is an entire function with $\varrho(k, \sum_{n=1}^{\infty} q_n) \leqslant \alpha$;*

(ii) *if $\varrho(k, \sum_{n=1}^{\infty} q_n) = \alpha$ then*

$$\sigma(k, \sum_{n=1}^{\infty} q_n) \leqslant \begin{cases} \dfrac{\beta}{e\alpha}, & k = 1; \\ \beta, & k \geqslant 2. \end{cases}$$

Proof. For $k=1$ see Lemma 3.3 in [5]. Suppose that k is not less than 2. At first we test the boundedness of the sets $\left\{\|q_n\|\left(\frac{r}{d}\right)^n : n\in N\right\}$ for $r>0$. By the assumption we have

$$\|q_n\|\left(\frac{r}{d}\right)^n \leqslant r^n\left(\frac{\beta}{\ln_{k-1} n}\right)^{n/\alpha}, \quad n>n_0,\ r>0.$$

Put

$$h_r(x) = r^x\left(\frac{\beta}{\ln_{k-1} x}\right)^{x/\alpha}, \quad x>n_0,\ r>0. \tag{2.2}$$

By the methods of the calculus we obtain that for sufficiently large r the maximum of the function h_r is reached for $x=x_r$, where x_r is determined by the equation

$$\alpha \ln r = \ln_k x_r - \ln\beta + \prod_{j=1}^{k-1}\frac{1}{\ln_j x_r}. \tag{2.3}$$

So there exist positive constants R and X such that for $r>R$ and $n>X$

$$\|q_n\|\left(\frac{r}{d}\right)^n \leqslant h_r(x_r). \tag{2.4}$$

On this account the set $\left\{\|q_n\|\left(\frac{r}{d}\right)^n : n\in N\right\}$ is bounded for $r>R$ and consequently for $r>0$, whence $\sum_{n=1}^{\infty} q_n$ is entire.

Observe that for each value of $r>0$ there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \|q_{\nu(r)}\|\left(\frac{r}{d}\right)^{\nu(r)}$$

and the equality does not hold for any $n>\nu(r)$. It is easy to see that $\nu(r)$ becomes infinite with r, so there exists an $R'>R$ such that $\nu(r)>X$ for $r>R'$ and we get, because of (2.4),

$$M^*(r) \leqslant h_r(x_r), \quad r>R'. \tag{2.5}$$

Combining (2.2), (2.3) and (2.5) we obtain

$$M^*(r) \leqslant \left\{\exp\left[\ln_k x_r - \ln\beta + \prod_{j=1}^{k-1}\frac{1}{\ln_j x_r}\right]\frac{\beta}{\ln_{k-1} x_r}\right\}^{x_r/\alpha}, \quad r>R'. \tag{2.6}$$

From (2.6) we get

$$\ln_{k+1} M^*(r) \leqslant \ln_k\left[\frac{x_r}{\alpha}\prod_{j=1}^{k-1}\frac{1}{\ln_j x_r}\right], \quad r>R'.$$

But x_r tends to infinity with r, so

$$\ln_k\left[\frac{x_r}{\alpha}\prod_{j=1}^{k-1}\frac{1}{\ln_j x_r}\right] \cong \ln_k x_r \cong \ln(\beta r^\alpha) \quad \text{for } r\to\infty.$$

Thus for every $\varepsilon>0$ there exists an R_ε such that for $r>R_\varepsilon$

$$\ln_{k+1} M^*(r) \leqslant (1+\varepsilon)\ln(\beta r^\alpha)$$

and consequently

$$\frac{\ln_{k+1} M^*(r)}{\ln r} \leqslant (1+\varepsilon)\left[\alpha+\frac{\ln\beta}{\ln r}\right], \qquad r>R_\varepsilon .$$

After passing to the upper limit we obtain

$$\varrho^*(k) \leqslant (1+\varepsilon)\alpha .$$

Since this inequality holds for every $\varepsilon>0$, we get $\varphi^*(k)\leqslant\alpha$ and by Lemma 2.2

$$\varphi(k)\leqslant\alpha .$$

The proof of (ii) proceeds on the same lines.

LEMMA 2.4. *Let $(q_n)_{n\in N}$ be a sequence of polynomials of the respective degrees not exceeding n. If there exist positive numbers α, β and n_0 such that for every $n>n_0$*

$$\|q_n\| \leqslant d^n\left[\exp\left(\frac{n}{\beta}\right)^{1/\alpha}\right]^{-n},$$

then

(i) $\sum_{n=1}^{\infty} q_n$ *is an entire function with* $\varrho_l(\sum_{n=1}^{\infty} q_n)\leqslant\alpha+1$;

(ii) *if* $\varrho_l(\sum_{n=1}^{\infty} q_n) = \alpha+1$, *then* $\sigma_l(\sum_{n=1}^{\infty} q_n)\leqslant\chi(\alpha)\beta$, *where* $\chi(\alpha) = \dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$.

Proof. By the assumption we have

$$\|q_n\|\left(\frac{r}{d}\right)^n \leqslant r^n\left[\exp\left(\frac{n}{\beta}\right)^{1/\alpha}\right]^{-n}, \qquad n>n_0 .$$

One can readily check that the maximum of the function $\left\{x\to r^x\left[\exp\left(\frac{x}{\beta}\right)^{1/\alpha}\right]^{-x}\right\}$ is $\exp[\chi(\alpha)\beta(\ln r)^{\alpha+1}]$ and it is reached for $x = x_r = \left(\frac{\alpha}{\alpha+1}\right)^\alpha\beta(\ln r)^\alpha$. Observe that x_r tends to infinity with r, so there exists an R such that $x_r>n_0$ for $r>R$. Thus

$$\|q_n\|\left(\frac{r}{d}\right)^n \leqslant \exp[\chi(\alpha)\beta(\ln r)^{\alpha+1}], \qquad r>R,\ n>n_0 .$$

Hence the set $\left\{\|q_n\|\left(\frac{r}{d}\right)^n : n\in N\right\}$ is bounded and the function $\sum_{n=1}^{\infty} q_n$ is entire. Analogously as in the proof of Lemma 2.3 we obtain for r sufficiently large

$$M^*(r)\leqslant\exp[\chi(\alpha)\beta(\ln r)^{\alpha+1}] .$$

From this inequality we get for large r

$$(\ln\ln r)^{-1}\ln\ln M^*(r)\leqslant(\ln\ln r)^{-1}\ln[\chi(\alpha)\beta(\ln r)^{\alpha+1}]$$

and

$$(\ln r)^{-(\alpha+1)}\ln M^*(r)\leqslant(\ln r)^{-(\alpha+1)}\chi(\alpha)\beta(\ln r)^{\alpha+1}\,.$$

After passing to the upper limit we obtain

$$\varrho_l^*(\sum_{n=1}^{\infty} q_n)\leqslant\alpha+1\,,$$

$$\sigma_l^*(\sum_{n=1}^{\infty} q_n)\leqslant\chi(\alpha)\beta, \quad \text{if } \varrho_l=\alpha+1\,.$$

Due to Lemma 2.2 the proof is completed.

3. Best approximation and interpolation. Let f be a function defined and bounded on K. Preserving the notations of the Section 1, put for $n\in N$

$$E_n^{(1)}=E_n^{(1)}(f,K)=E_n(f,K)\,,$$
$$E_n^{(2)}=E_n^{(2)}(f,K)=||f-l_n||\,,$$
$$E_n^{(3)}=E_n^{(3)}(f,K)=||l_n-l_{n-1}||\,.$$

LEMMA 3.1. ([5], Lemma 3.2). *For every positive integer* n

(i) $$E_n^{(1)}\leqslant E_n^{(2)}\leqslant(n+2)E_n^{(1)}\,,$$

(ii) $$E_{n+1}^{(3)}\leqslant 2(n+3)E_n^{(1)}\,.$$

Proposition 3.1. *The function f has an entire continuation if and only if*

$$\lim_{n\to\infty}(E_n^{(j)})^{1/n}=0, \quad j=1,2,3\,.$$

Proof. Necessity. For $j=1,2$ — see [4]. For $j=3$ — use Lemma 3.1

Sufficiency. For $j=1,2$ — see [4]. For $j=3$ the function $\tilde{f}=l_1+\sum_{n=2}^{\infty}(l_n-l_{n-1})$ gives the required continuation of f.

4. Growth of entire function by means of approximation. Let g be an entire function. In this section $\varrho(k)$, $\sigma(k)$, ϱ_l, σ_l and $\tilde{M}(r)$ stand for $\varrho(k,g)$, $\sigma(k,g)$, $\varrho_l(g)$, $\sigma_l(g)$ and $\tilde{M}(r,g)$ respectively, $E_n^{(j)}$ denotes $E_n^{(j)}(g|_K,K)$. Likewise to definition of $\ln_k$ we put for $k\in N$ $\exp_k x=\exp(\exp_{k-1}x)$, $\exp_0 x=x$.

LEMMA 4.1. *If there exist positive numbers R_0, τ, μ and $k\in N$ such that for every $r>R_0$*

$$\tilde{M}(r)\leqslant\exp_k(\tau r^{\mu})\,,$$

then

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\mu}(E_n^{(j)})^{1/n}\leqslant\begin{cases}d(e\mu\tau)^{1/\mu}, & k=1,\\ d\tau^{1/\mu}, & k\geqslant 2,\end{cases}\quad j=1,2,3.$$

Proof. T. Winiarski has proved ([5], p. 266) that there exists a positive number A, depending only on K and g, such that for every $\varepsilon>0$

$$E_n^{(2)}\leqslant A(de^{\varepsilon})^n\tilde{M}(r)r^{-n} \tag{4.1}$$

— for r and n sufficiently large, $r>R_\varepsilon$ and $n>N_\varepsilon$ say. Hence and by the assumption we obtain that

$$E_n^{(2)}\leqslant A(de^{\varepsilon})^n\exp_k(\tau r^{\mu})r^{-n}\quad \text{for } r>R \text{ and } n>N_\varepsilon, \tag{4.2}$$

where $R=R_0+R_\varepsilon$.

Put

$$u(r)=\mu\prod_{j=0}^{k-1}\exp_j(\tau r^{\mu}),\qquad r>0.$$

Since u is strictly monotonic, there exists its inverse, u^{-1}. Set $r_n=u^{-1}(n)$. Let N be such a number that $N\geqslant N_\varepsilon$ and $r_n>R$ for $n>N$. From (4.2) we obtain

$$E_n^{(2)}\leqslant A(de^{\varepsilon})^n\exp_k(\tau r_n^{\mu})r_n^{-n},\qquad n>N.$$

Hence

$$(\ln_{k-1}n)^{1/\mu}(E_n^{(2)})^{1/n}\leqslant A^{1/n}de^{\varepsilon}[\exp_k(\tau r_n^{\mu})]^{1/n}(\ln_{k-1}n)^{1/\mu}r_n^{-1},\qquad n>N. \tag{4.3}$$

Observe that

$$\lim_{n\to\infty}[\exp_k(\tau r_n^{\mu})]^{1/n}=\begin{cases}e^{1/\mu}, & k=1,\\ 1, & k\geqslant 2;\end{cases}$$

$$\lim_{n\to\infty}(\ln_{k-1}n)^{1/\mu}r_n^{-1}=\begin{cases}(\mu\tau)^{1/\mu}, & k=1,\\ \tau^{1/\mu}, & k\geqslant 2.\end{cases}$$

Therefore, after passing to the upper limit in (4.3) we get

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\mu}(E_n^{(2)})^{1/n}\leqslant\begin{cases}de^{\varepsilon}(e\mu\tau)^{1/\mu}, & k=1,\\ de^{\varepsilon}\tau^{1/\mu}, & k\geqslant 2.\end{cases}$$

On account of arbitrariness of ε the factor e^{ε} can be omitted. Hence and by Lemma 3.1

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\mu}(E_n^{(j)})^{1/n}\leqslant\begin{cases}d(e\mu\tau)^{1/\mu}, & k=1,\\ d\tau^{1/\mu}, & k\geqslant 2,\end{cases}$$

as asserted.

LEMMA 4.2. *If there exist positive numbers R, τ and μ such that for $r>R$*

$$\tilde{M}(r)\leqslant\exp[\tau(\ln r)^{\mu+1}],$$

then

$$\limsup_{n\to\infty} \frac{n^{1/\mu}}{\ln \dfrac{d}{(E_n^{(j)})^{1/n}}} \leqslant \left(\frac{\tau}{\chi(\mu)}\right)^{1/\mu}, \quad j = 1, 2, 3, \tag{4.4}$$

(*we recall that* $\chi(\mu) = \dfrac{\mu^\mu}{(\mu+1)^{\mu+1}}$).

Proof. From (4.1) and by the assumption we obtain that for each $\varepsilon>0$ there exist R_ε and N_ε such that

$$E_n^{(2)} \leqslant A(de^\varepsilon)^n \exp[\tau(\ln r)^{\mu+1}] r^{-n}, \quad r>R_\varepsilon,\ n>N_\varepsilon \tag{4.5}$$

(A being a constant).

Let $N'_\varepsilon > N_\varepsilon$ be so large that

$$\exp\left[\frac{n}{\tau(\mu+1)}\right]^{1/\mu} > R_\varepsilon \quad \text{for } n>N'_\varepsilon .$$

If we put in (4.5) $r_n = \exp\left(\dfrac{n}{\tau(\mu+1)}\right)^{1/\mu}$, then we obtain

$$E_n^{(2)} \leqslant A(de^\varepsilon)^n \left[\exp\left(\frac{n}{\tau(\mu+1)}\right)^{1/\mu}\right]^{-\frac{n\mu}{\mu+1}} .$$

Then

$$\frac{n^{1/\mu}}{\ln \dfrac{d}{(E_n^{(2)})^{1/n}}} \leqslant \frac{1}{\left(\dfrac{\mu}{\mu+1}\right)[\tau(\mu+1)]^{-1/\mu} - \dfrac{\ln(\varepsilon a^{1/n})}{n^{1/\mu}}} .$$

After passing to the upper limit we obtain (4.4) for $j = 2$. Due to Lemma 3.1 we have (4.4) also for $j = 1, 3$.

THEOREM 4.1. *Let g be an entire transcendental function of index $Ig \leqslant k$, k being a positive constant. Then*

$$\varrho(k) = \limsup_{n\to\infty} \frac{\ln_k n}{\ln \dfrac{d}{(E_n^{(j)})^{1/n}}}, \quad j = 1, 2, 3. \tag{4.6}$$

Proof. Denote the right-hand member of (4.6) by γ_j ($j = 1, 2, 3$). We start by showing that $\varrho(k) \leqslant \gamma_3$. Suppose that γ_3 is finite. Then for every $\varepsilon>0$ we can find an N_ε such that

$$\frac{\ln_k n}{\ln \dfrac{d}{(E_n^{(3)})^{1/n}}} \leqslant \gamma_3 + \varepsilon, \quad n>N_\varepsilon ,$$

whence

$$E_n^{(3)} \leqslant d^n \left(\frac{1}{\ln_{k-1} n} \right)^{\frac{n}{\gamma_3+\varepsilon}} .$$

Then, because of Lemma 2.3 $\varrho(k) \leqslant \gamma_3 + \varepsilon$. Hence $\varrho(k) \leqslant \gamma_3$ provided γ_3 is finite. But this inequality is evidently true if $\gamma_3 = +\infty$, so $\varrho(k) \leqslant \gamma_3$ regardless of the value of γ_3.

Moreover, by Lemma 3.1 we have $\gamma_3 \leqslant \gamma_1 \leqslant \gamma_2$, whence $\varrho(k) \leqslant \gamma_j$, $j = 1, 2, 3$.

On the other hand, by Lemma 2.1 we have

$$\varrho(k) = \limsup_{r \to \infty} (\ln r)^{-1} \ln_{k+1} \tilde{M}(r) .$$

So, given $\mu > \varrho(k)$, we can find an R_μ such that

$$(\ln r)^{-1} \ln_{k+1} \tilde{M}(r) \leqslant \mu, \quad r > R_\mu .$$

Then

$$\tilde{M}(r) \leqslant \exp_k(r^\mu) .$$

Therefore by Lemma 4.1

$$\limsup_{n \to \infty} (\ln_{k-1} n)^{1/\mu} (E_n^{(j)})^{1/n}$$

is finite. So there exists a positive constant B such that for every n

$$(\ln_{k-1} n)^{1/\mu} (E_n^{(j)})^{1/n} \leqslant Bd, \quad j = 1, 2, 3 .$$

Hence

$$\frac{\ln_k n}{\ln \dfrac{d}{(E_n^{(j)})^{1/n}}} \leqslant \mu + \mu \frac{\ln B}{\ln \dfrac{d}{(E_n^{(j)})^{1/n}}} .$$

After passing to the upper limit we obtain

$$\gamma_j \leqslant \mu, \quad j = 1, 2, 3 .$$

Since this inequality holds for every $\mu > \varrho(k)$, we have

$$\gamma_j \leqslant \varrho(k) .$$

The proof is completed.

THEOREM 4.2. *Let g be an entire transcendental function with $\varrho(k) > 0$ and finite, k being a positive integer. Then*

$$\limsup_{n \to \infty} (\ln_{k-1} n)^{1/\varrho(k)} (E_n^{(j)})^{1/n} = \begin{cases} d\big(e\varrho(1)\sigma(1)\big)^{1/\varrho(1)}, & k = 1 , \\ d\sigma(k)^{1/\varrho(k)}, & k \geqslant 2 , \end{cases} \quad j = 1, 2, 3 .$$

Proof. Suppose that k is not less than 2 (for $k = 1$ the proof proceeds on the same lines). Given $\tau > \sigma(k)$, we can find an R_τ such that

$$\tilde{M}(r) \leqslant \exp_k(\tau r^{\varrho(k)}), \quad r > R_\tau .$$

Then by Lemma 4.1

$$\limsup_{n\to\infty} (\ln_{k-1} n)^{1/\varrho(k)} (E_n^{(j)})^{1/n} \leqslant d\tau^{1/\varrho(k)}, \quad j = 1, 2, 3,$$

and on account of the arbitrariness of $\tau > \sigma(k)$

(4.7) $$\limsup_{n\to\infty} (\ln_{k-1} n)^{1/\varrho(k)} (E_n^{(j)})^{1/n} \leqslant d\sigma(k)^{1/\varrho(k)}, \quad j = 1, 2, 3.$$

Suppose that this inequality is sharp for $j = 3$. Then we can find $\vartheta < \sigma(k)$ such that for n sufficiently large

$$(\ln_{k-1} n)^{1/\varrho(k)} (E_n^{(3)})^{1/n} \leqslant d\vartheta^{1/\varrho(k)}.$$

This inequality can be rewritten in the form

$$E_n^{(3)} \leqslant d^n \left(\frac{\vartheta}{\ln_{k-1} n} \right)^{n/\varrho(k)}.$$

Therefore by Lemma 2.3 $\sigma(k) \leqslant \vartheta$ and we get a contradiction, because ϑ has been chosen less than $\sigma(k)$. Hence and by Lemma 3.1 we obtain that in (4.7) actually holds equality, as asserted.

In the same way as above, using Lemmas 2.4 and 4.2 instead of Lemmas 2.3 and 4.1 we can prove the following two theorems.

THEOREM 4.3. *Let g be an entire transcendental function. Then*

$$\varrho_l = 1 + \limsup_{n\to\infty} \frac{\ln n}{\ln \ln \dfrac{d}{(E_n^{(j)})^{1/n}}}, \quad j = 1, 2, 3.$$

THEOREM 4.4. *Let g be an entire transcendental function with $\varrho_l > 1$ and finite. Then*

$$\limsup_{n\to\infty} \frac{n^{1/\alpha}}{\ln \dfrac{d}{(E_n^{(j)})^{1/n}}} = \left(\frac{\sigma_l}{\chi(\alpha)} \right)^{1/\alpha}, \quad \textit{where } \alpha = \varrho_l - 1.$$

5. Growth of entire continuation. Let f be a function defined and bounded on K and let k be a fixed positive integer. Put $E_n = E_n(f, K)$.

THEOREM 5.1. *If*

$$\alpha = \limsup_{n\to\infty} \frac{\ln_k n}{\ln \dfrac{d}{E_n^{1/n}}}$$

is finite, then the function $\tilde{f} = t_1 + \sum_{n=2}^{\infty} (t_n - t_{n-1})$ *is entire,* $\tilde{f}(z) = f(z)$ *for* $z \in K$ *and* $\varrho(k, \tilde{f}) = \alpha$.

Proof. Given $\mu>\alpha$, then there exists an N_μ such that

$$\frac{\ln_k n}{\ln \frac{d}{E_n^{1/n}}} \leqslant \mu, \quad n \geqslant N_\mu .$$

Hence

$$E_n \leqslant d^n \left(\frac{1}{\ln_{k-1} n}\right)^{n/\mu}, \quad n \geqslant N_\mu .$$

It implies that for n sufficiently large

$$\|t_n - t_{n-1}\| \leqslant d^n \left(\frac{2}{\ln_{k-1} n}\right)^{n/\mu} .$$

Therefore by Lemma 2.3 the function $\tilde{f}$ is entire and the index of $\tilde{f}$ satisfies $I\tilde{f} \leqslant k$. So by Theorem 4.1 $\varrho(k, \tilde{f}) = \alpha$ and the proof is completed.

Theorem 1.1 (i) follows immediately from Theorems 4.1 and 5.1.

Denote

$$\Gamma_k = \{\gamma \in (0, +\infty) \colon \limsup_{n\to\infty} (\ln_{k-1} n)^{1/\gamma} E_n^{1/n} = 0\}$$

THEOREM 5.2. *If $\Gamma_k \neq \emptyset$, then the function $\tilde{f} = t_1 + \sum_{n=2}^{\infty} (t_n - t_{n-1})$ is entire, $\tilde{f}(z) = f(z)$ for $z \in K$ and $\varrho(k, \tilde{f}) = \alpha$, where $\alpha = \inf \Gamma_k$. If additionaly $\alpha>0$, then*

$$\sigma(k, \tilde{f}) = \begin{cases} \dfrac{1}{e\alpha}\left(\dfrac{\beta}{d}\right)^{\alpha}, & k = 1, \\ \left(\dfrac{\beta}{d}\right)^{\alpha}, & k \geqslant 2, \end{cases}$$

where $\beta = \limsup_{n\to\infty} (\ln_{k-1} n)^{1/\alpha} \cdot E_n^{1/n}$.

Proof. Given $\gamma \in \Gamma_k$, we can find a real number B such that for every n

$$(\ln_{k-1} n)^{1/\gamma} E_n^{1/n} \leqslant B .$$

Then

$$E_n \leqslant B^n (\ln_{k-1} n)^{-n/\gamma}, \quad n \in N .$$

From this we obtain for n sufficiently large

$$\|t_n - t_{n-1}\| \leqslant d^n \left(\frac{\mu}{\ln_{k-1} n}\right)^{n/\gamma}, \quad \text{where } \mu = 2\left(\frac{B}{d}\right)^{\gamma} .$$

Therefore by Lemma 2.3 the function $\tilde{f}$ is entire and $\varrho(k, \tilde{f}) \leqslant \gamma$. Hence $\varrho(k, \tilde{f}) \leqslant \alpha = \inf \Gamma_k$.

Because $\varrho(k,\tilde{f})$ is finite, we can apply Theorem 4.1 and we obtain

$$\varrho(k,\tilde{f})=\limsup_{n\to\infty}\frac{\ln_k n}{\ln\dfrac{d}{E_n^{1/n}}}.$$

Given $\gamma>\varrho(k,\tilde{f})$, take γ' less than γ, but greater than $\varrho(k,\tilde{f})$. Then there exists an $N_{\gamma'}$ such that

$$\frac{\ln_k n}{\ln\dfrac{d}{E_n^{1/n}}}\leqslant\gamma',\qquad n\geqslant N_{\gamma'}.$$

From this we get for $n\geqslant N_{\gamma'}$

$$(\ln_{k-1}n)^{1/\gamma'}E_n^{1/n}\leqslant d,$$

whence

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\gamma}E_n^{1/n}=0.$$

Thus we obtain that for every $\gamma>\varrho(k,\tilde{f})$, γ belongs to Γ_k, so $\varrho(k,\tilde{f})\geqslant\alpha=\inf\Gamma_k$. Since the opposite inequality holds, we conclude that $\varrho(k,\tilde{f})=\alpha$ as asserted.

Suppose that $\alpha>0$ and $\beta<+\infty$. Then there exists an n_0 such that

$$E_n\leqslant(2\beta)^n(\ln_{k-1}n)^{-n/\alpha},\qquad n>n_0,$$

whence

$$\|t_n-t_{n-1}\|\leqslant d^n\left[\frac{\left(\dfrac{3\beta}{d}\right)^\alpha}{\ln_{k-1}n}\right]^{n/\alpha},$$

for n sufficiently large. Then by Lemma 2.3 $\sigma(k,\tilde{f})$ is finite. Thus, due to Theorem 4.2

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\alpha}E_n^{1/n}=\begin{cases}d(e\alpha\sigma(1,\tilde{f}))^{1/\alpha}, & k=1,\\ d\sigma(k,\tilde{f})^{1/\alpha}, & k\geqslant 2.\end{cases}$$

So

$$\sigma(k,\tilde{f})=\begin{cases}\dfrac{1}{e\alpha}\left(\dfrac{\beta}{d}\right)^\alpha, & k=1,\\ \left(\dfrac{\beta}{d}\right)^\alpha, & k\geqslant 2,\end{cases}$$

provided β is finite. Evidently $\sigma(k,\tilde{f})$ is infinite if β is, so the proof is completed.

Observe that if

$$\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\alpha}E_n^{1/n}$$

is positive and finite, then

(i) for every $\gamma>\alpha$: $\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\gamma}E_n^{1/n} = 0$,

(ii) for every $\vartheta<\alpha$: $\limsup_{n\to\infty}(\ln_{k-1}n)^{1/\vartheta}E_n^{1/n} = +\infty$, so α is the infimum of Γ_k. Now from Theorems 4.2, 5.2 and the above remark we obtain Theorem 1.1 (ii).

If we deal with ϱ_l and σ_l, then we can state the following two theorems.

THEOREM 5.3. *If*

$$\alpha = \limsup_{n\to\infty}\frac{\ln n}{\ln\ln\dfrac{d}{E_n^{1/n}}}$$

is finite, then the function $\tilde{f} = t_1+\sum_{n=2}^{\infty}(t_n-t_{n-1})$ *is entire,* $\tilde{f}(z) = f(z)$ *for* $z\in K$ *and* $\varrho_l(\tilde{f}) = \alpha+1$.

Denote

$$\Omega = \left\{\omega\in(0,+\infty):\ \limsup_{n\to\infty}\frac{n^{1/\omega}}{\ln\dfrac{d}{E_n^{1/n}}} = 0\right\}.$$

THEOREM 5.4. *If* $\Omega\neq\emptyset$, *then the function* $\tilde{f} = t_1+\sum_{n=2}^{\infty}(t_n-t_{n-1})$ *is entire,* $\tilde{f}(z) = f(z)$ *for* $z\in K$ *and*

$$\varrho_l(\tilde{f}) = \alpha+1, \quad \textit{where } \alpha = \inf\Omega.$$

If additionally $\alpha>0$, *then*

$$\sigma_l(\tilde{f}) = \chi(\alpha)\beta^{\alpha},$$

where $\beta = \limsup_{n\to\infty}\dfrac{n^{1/\alpha}}{\ln\dfrac{d}{E_n^{1/n}}}$.

We omit the proofs of Theorems 5.3 and 5.4 because they are quite analogous to those of Theorems 5.1 and 5.2 respectively.

As a simple consequence of the above two theorems and Theorems 4.3 and 4.4 we obtain Theorem 1.1 (iii) and (iv).

Remark. It is obvious that in this section we may replace E_n by $E_n^{(j)}$ for $j = 2, 3$.

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