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On the differentiability of generalized subadditive functions

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1. In this paper we shall deal with the functional inequality

$$\psi(x+y) \leqslant F(\psi(x), \psi(y)) \tag{1}$$

and the associated equation

$$\varphi(x+y) = F(\varphi(x), \varphi(y)), \tag{2}$$

where $F\colon J^2 \to J$ is a given function, $J = (a, b)$ is an interval (we admit $a = -\infty$ and/or $b = +\infty$), φ, ψ are unknown functions defined on R or R^n ($n \geqslant 2$).

We investigate differentiable solutions of (1) and (2). For continuous solutions of equation (2) the following results are known (see [1], 54–58).

Lemma 1.

(a) *Equation* (2) *has in R a nonconstant and continuous solution iff the function* $F\colon J^2 \to J$ *is continuous with respect to each variable and the structure* (J, F) *is a group.*

(b) *If equation* (2) *has in R nonconstant and continuous solution, then the function* $F\colon J^2 \to J$ *is strictly increasing with respect to each variable.*

(c) *If a function* $f\colon R \to J$ *is continuous, nonconstant solution of equation* (2), *then*

$$f(0) = e,$$

where e is the neutral element of the group (J, F), *and the general continuous solution* $\varphi\colon R \to J$ *of equation* (2) *is given by the formula*

$$\varphi(x) = f(cx), \quad x \in R, \tag{3}$$

where c is a real number. All these solutions are strictly monotonic in R.

According to Lemma 1 we shall assume that

(H) *Equation* (2) *has a continuous and strictly increasing solution* $f\colon R \to J$.

In the sequel f will denote the solution of (2) occurring in (H). Theorems given in [2] (for $n = 1$) and [3] imply the form of continuous solutions of inequality (1) in R^n:

THEOREM 1. *Let hypothesis* (H) *be fulfilled. All the continuous solutions* $\psi\colon R^n \to J$, $n \geqslant 1$, *of inequality* (1) *are given by the formula*

$$\psi(x) = f(h(x)), \quad x \in R^n, \tag{4}$$

where $h\colon R^n \to R$ *is an arbitrary continuous, subadditive function,* i.e.

$$h(x+y) \leqslant h(x)+h(y), \quad x, y \in R^n. \tag{5}$$

In the paper we shall make use of the results on the differentiability of subadditive functions and we shall prove some similar results for solutions of inequality (1). Lemmas given below contain results from [4], p. 251 (Lemma 2) and [5] (Lemma 3).

LEMMA 2. *If* $h\colon R \to R$ *is a subadditive function differentiable at* $x = 0$ *and such that* $h(0) = 0$, *then* h *is a continuous linear function,* i.e.

$$h(x) = cx, \; x \in R, \quad c = h'(0). \tag{6}$$

LEMMA 3. *Let* $h\colon R^n \to R$ *be an arbitrary subadditive function and put* $h_i\colon R \to R$, $h_i(t) = h(0, \ldots 0, t, 0, \ldots, 0)$ (*with* t *on the i-th place*) *and*

$$A_i = \inf\{h_i(t)/t, \, t<0\} \quad \text{and} \quad B_i = \sup\{h_i(t)/t, \, t>0\}; \tag{7}$$

for $i = 1, 2, \ldots, n$. If $A_i = B_i$ *then* h *is a continuous solution of the Cauchy functional equation in* R^n, *of the form*

$$h(x_1, \ldots, x_n) = A_1 x_1 + \ldots + A_n x_n, \quad (x_1, \ldots, x_n) \in R^n. \tag{8}$$

In the last section of the paper we obtain a result generalizing one due to J. E. Wetzel [5] and concerning a special case of inequality (1).

2. We start with

THEOREM 1. *If* (H) *is assumed, then the solution* f *of equation* (2) *is differentiable in* R *iff it is differentiable at zero, for each* x *there exists the derivative of function* F *with respect to the second variable in the set* $J \times \{e\}$ *and*

$$f'(x) = f'(0) F'_v(f(x), e), \quad x \in R. \tag{9}$$

Proof. Sufficiency. We shall prove that f is differentiable at an arbitrary $x \in R$. Since f fulfils equation (2) in R, then we have

$$f(x+h)-f(x) = F(f(x), f(h))-f(x) = F(f(x), f(h))-F(f(x), e)$$

(note that e is the neutral element of the group (J, F)). Hence, by virtue of $f(0) = e$ (Lemma 1) we have

$$\frac{f(x+h)-f(x)}{h} = \frac{F(f(x), f(h))-F(f(x), f(0))}{f(h)-f(0)} \, \frac{f(h)-f(0)}{h} \tag{10}$$

It follows from differentiability of f at zero and existence of $F'_v(f(x), e)$ that there exists the limit of the right hand side of (10) as h tends to zero. Therefore f is differentiable in R and the derivative of f is given by (9).

Necessity. Now let f be differentiable in R. Then there exists the limit at the left hand side of equality (10) as h tends to zero and the limit of the second factor of the right hand side of (10) equals $f'(0)$.

If $f'(0) = 0$, then $f'(x) = 0$ in R which contradicts (H). Therefore (10) implies the existence of the derivative which we denoted by $F'_v(f(x), e)$ which complete the proof of theorem.

Remark. It follows from (9) that if the function f defined by (H) is differentiable in R, then

(11) $$f'(0) \neq 0$$

and

(12) $$F'_v(e, e) = 1 .$$

Now we shall prove a result concerning inequality (1) which is similar to Lemma 2. Under a stronger assumption it was proved by D. Brydak [2].

THEOREM 3. *If f defined by* (H) *is differentiable in R, then every solution ψ of inequality* (1) *differentiable at zero, continuous in R and fulfilling the condition*

(13) $$\psi(0) = f(0)$$

is a solution of equation (2) *in R.*

Proof. Let ψ be a solution of inequality (1) with properties mentioned in the assumptions of the theorem. We have by virtue of Theorem 1

(14) $$\psi(x) = f(h(x)), \quad x \in R,$$

where h is a continuous, subadditive function. Hence

(15) $$h(x) = f^{-1}(\psi(x)), \quad x \in R.$$

Putting $x = 0$ in (15) we have from (13) $h(0) = 0$, as $f(0) = e$ (Lemma 1). It follows from (11) that the derivative of inverse function f^{-1} exists at the point e. The function ψ is differentiable at zero. Hence and from (15) we have the differentiability of function h in zero. Therefore Lemma 2 implies formula (6) and it follows from (14), (3) and Lemma 1 that ψ is a solution of equation (2).

3. A generalization of Theorem 3 and Lemma 3 is contained in

THEOREM 4. *Let the f defined by* (H) *be differentiable in R. If $\psi\colon R^n \to (a, b)$ is differentiable at the point* $(0, \ldots, 0) \in R^n$, *continuous in R^n and satisfies inequality* (1) *and the condition*

(16) $$\psi(0, \ldots, 0) = f(0),$$

then ψ is solution of equation (2) *in R^n.*

Proof. Let $\psi: R^n \to (a, b)$ be a solution of inequality (1) fulfilling the assumptions of the theorem. Then Theorem 1 implies that

$$\psi(x_1, \dots, x_n) = f(h(x_1, \dots, x_n)), \quad (x_1, \dots, x_n) \in R^n, \tag{17}$$

where $h: R^n \to R$ is a continuous, subadditive function.
Putting $x_j = 0$ for $i \neq j$ in (17) and denoting $\psi_i(x_i) = (0, \dots, 0, x_i, 0, \dots, 0)$ (similarly as h_i in Lemma 3) we obtain

$$\psi_i(x_i) = f(h_i(x_i)) \quad \text{for } i = 1, 2, \dots, n. \tag{18}$$

The differentiability of ψ at $(0, \dots, 0)$ implies the differentiability of ψ_i at zero for $i = 1, 2, \dots, n$. Hence, by virtue of the assumptions of the theorem and Theorem 3 we infer that ψ_i fulfils equation (2) in R and from Lemma 1(d) we have

$$\psi_i(x_i) = f(c_i x_i) \quad \text{for } i = 1, 2, \dots, n. \tag{19}$$

From (18), (19) and the monotonicity of f in R we infer that

$$h_i(x_i) = c_i x_i \quad \text{for } i = 1, 2, \dots, n.$$

Hence by definition (7) we have

$$A_i = B_i = c_i \quad \text{for } i = 1, 2, \dots, n$$

then by Lemma 3, function h in formula (17) has the form (8). Therefore

$$\psi(x_1, \dots, x_n) = f(c_1 x_1 + \dots + c_n x_n)$$

and by virtue of [1] it fulfils equation (2).

4. Now we shall consider the problem under what conditions a solution of inequality (1) bounded by a differentiable function is differentiable itself. For the inequality

$$\psi(x+y) \geqslant \psi(x)\psi(y)$$

an answer to this question was given by Wetzel in [5]. For inequality (1) in R we have

THEOREM 5. *Suppose that f defined by* (H) *is differentiable in R and that $F: J^2 \to J$ is differentiable with respect to the second variable in J and its derivative F'_v is a function continuous with respect to each variable (separately) in J. If ψ is a continuous solution of inequality* (1) *in R bounded from above by a function $g: R \to J$ differentiable in a neighbourhood of zero and satisfying*

$$\psi(0) = g(0) = e, \tag{20}$$

then ψ is differentiable in R and

$$\psi(x) = f(cx), \quad x \in R, \quad c = g'(0)/f'(0).$$

Proof. Let ψ be a continuous solution of inequality (1) in R and g — a function fulfilling the assumptions of the theorem. Then condition (20) is fulfilled and

$$\psi(x) \leqslant g(x) \quad x \in R. \tag{21}$$

First we shall prove that ψ is differentiable at zero. From (20) and (21) we obtain

$$\psi(h) - \psi(0) \leqslant g(h) - g(0). \tag{22}$$

On the other hand, (1), (21) and the monotonicity of F (Lemma 1) imply

$$\psi(0) = \psi(h-h) \leqslant F(\psi(h), \psi(-h)) \leqslant F(\psi(h), g(-h)) \tag{23}$$

Multiplying (23) by -1, adding $\psi(h)$ to its both sides we have by virtue of (20) (note that e is the neutral element of F)

$$\psi(h) - \psi(0) \geqslant F(\psi(h), g(0)) - F(\psi(h), g(-h)). \tag{24}$$

Applying the mean value theorem to the right hand side of (23) we obtain from (22) and (24) the inequalities

$$F'_v(\psi(h), \gamma(h))(g(0) - g(-h)) \leqslant \psi(h) - \psi(0) \leqslant g(h) - g(0) \tag{25}$$

where $\gamma(h)$ lies between $g(0)$ and $g(-h)$, so that it tends to $g(0)$ when $h \to 0$. Divide now the sides of (25) by $h > 0$ and make use of the assumed regularity of the functions F, g, ψ and F'_v to obtain for $h \to 0$

$$g'(0) F'_v(\psi(0), g(0)) \leqslant \psi'(0) \leqslant g'(0).$$

Dividing (25) by $h < 0$ and passing to the limit we get the above inequalities with the signs $\leqslant$ replaced by $\geqslant$. This, together with (20) and (12) yields

$$\psi'(0) = g'(0) \tag{26}$$

i.e. the function ψ is differentiable at zero. On account of Theorem 1 the ψ supplies then a solution to the equation (2) and can be written in form (3), which implies the differentiability of ψ in the whole R. Taking derivatives of (3) and putting $x = 0$ we get $c = g'(0)/f'(0)$ and the proof of the theorem is completed.

Remark. It follows from the condition (26) that every majorant g of a function ψ in R which is differentiable in a neighbourhood of zero and takes on the value $\psi(0)$ at zero has to fulfill the condition $g'(0) = \psi'(0)$.

We conclude the paper by an example showing the role of the assumptions of theorems involving the function g.

Example. Let we consider inequality (5) i.e. (1) with the function $F(u, v) = u + v$. It fulfils all assumptions of Theorem 5 in R^2. The function $\psi(x) = |x|$ is a solution of inequality (5) in R for which there does not exist any majorant g fulfilling condition (20) as regular, as it is required in the theorem.

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