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Invariance of some polynomial conditions for compact subsets of C^N under holomorphic mappings

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Abstract. We show that the property for a pair (E, μ), where E is a compact subset of C^N and μ is a positive measure on E, to satisfy polynomial conditions of the type of Leja's famous polynomial condition is invariant under a large class of holomorphic mappings (with values in C^M, where $M \leqslant N$) containing, in particular, all open holomorphic mappings. This yields some new examples of the pairs (E, μ) satisfying the conditions under consideration.

1. Introduction. Let E be a Borel subset of the space C^N of N complex variables and let μ be a measure on E (throughout the paper by a measure we shall mean any nonnegative set function μ defined on Borel subsets of E and such that $\mu(\emptyset) = 0$). The pair (E, μ) is said to *satisfy condition* (L^*) if for every subset $\mathscr{F}$ of the space $P(C^N, C^1)$ of all polynomials from C^N to C^1, such that

$$\sup_{f \in \mathscr{F}} |f(z)| < \infty$$

μ-almost everywhere on E, and for every $b > 1$ there exist an open neighborhood G of $\bar{E}$, the closure of E, and a positive constant M such that for each $f \in \mathscr{F}$ we have

$$\|f\|_G \leqslant M b^{\deg f},$$

where $\|f\|_G = \sup_{z \in G} |f(z)|$, and $\deg f$ denotes the degree of the polynomial f.

The condition (L^*) plays an important role in the theory of the polynomial approximation (see [5], [8], [11]). Some examples of pairs (E, μ) satisfying (L^*) were listed in [5] and [11]. It is worth while to notice that the first example of such a pair was given by Leja [4] and is the following.

Example 1.1. If E is a Jordan rectifiable arc in C^1 and μ is the length measure on E, then $(E, \mu) \in (L^*)$. In particular, E can be thought of as a compact interval in R^1, the space of one real variable, and $\mu = \lambda_1$, the Lebesgue linear measure.

It is seen that

1.2. If $(E, \mu) \in (L^*)$, then E satisfies the following

CONDITION (B_0). *For every $b > 1$ there exists an open neighborhood G of $\bar{E}$ and a constant $M > 0$ such that for every polynomial $f \in P(C^N, C^1)$*

$$\|f\|_G \leqslant M b^{\deg f} \max(1, \|f\|_E).$$

If E is compact, then condition (B_0) is equivalent to the *L-regularity* of E (see Section 2). It was proved in [6] that if E is an L-regular polynomially convex compact set in C^N and h is an open holomorphic mapping defined in an open neighborhood U of E with values in C^M $(M \leqslant N)$, then $h(E)$ is also L-regular. Actually, the assumptions on h can be essentially weakened (see [6]). In this paper we show (Theorems 3.4 and 5.2) that a similar invariance property holds true for the condition (L^*) as well as for a weaker condition (L_0^*) defined in Section 5.

In Section 4 some specifications of Theorem 3.4 are given. In particular, we study there the case where $\mu = \lambda_{2N}$, the $2N$-dimensional Lebesgue measure.

The results of the paper permit to give some new examples of pairs (E, μ) satisfying (L^*) and (L_0^*) (see Section 6). They also answer a question posed by Siciak (personal communication). The new examples are preceded by a criterion of the *L^*-regularity* (with respect to the Lebesgue measure) which is an analogue of a criterion of the L-regularity due to Gončar (see [7]).

2. The L-regularity. Denote by $\mathrm{Psh}(C^N)$ the set of all plurisubharmonic functions in C^N and consider the set

$$L = \{u \in \mathrm{Psh}\, C^N\colon\ u(z) \leqslant m + \log(1+|z|),\ z \in C^N\},$$

m being a real constant that may depend on u, and $|z| = \max\{|z_j|\colon\ 1 \leqslant j \leqslant N\}$ for $z = (z_1, \ldots, z_N) \in C^N$. Given any subset E of C^N we define the *L-extremal function* associated with E as

$$V_E(z) = \sup\{u(z)\colon\ u \in L,\ u \leqslant 0 \text{ on } E\}$$

(see [10], [12]). If the function V_E is continuous in C^N, the set E is called *L-regular*. Let V_E^* denotes the upper regularization of V_E,

$$V_E^*(z) = \limsup_{w \to z} V_E(w), \quad z \in C^N,$$

and let

$$c(E) = \limsup_{|z| \to \infty} |z| \exp[-V_E^*(z)].$$

The number $c(E)$ is called the *L-capacity* of E (see [10]). If $N = 1$, $c(E)$ is known to be equal to the logarithmic capacity of E. We have

LEMMA 2.1 ([10], Theorem 3.6 and Corollary 3.9). *A countable union of L-polar sets, that is, by definition, sets with the L-capacity equal to zero, is L-polar.*

If E is compact, it is known ([10], [12]) that the function $\exp V_E$ is equal to Siciak's *extremal function* associated with E,

$$\Phi_E(z) = \sup[|p(z)|^{1/\deg p}\colon\ p \in P(C^N, C^1),\ \|p\|_E \leqslant 1,\ \deg p \geqslant 1\}, \quad z \in C^N.$$

Moreover, in this case, for the L-regularity of E it suffices that V_E be continuous at every point of E ([10], [12]). Hence by 1.2 we have

2.2. If E is compact and (E, μ) satisfies (L^*), then E is L-regular. In particular, $c(E)>0$.

Question 2.3. Is it true that for any Borel subset E of $\boldsymbol{C}^N$ and any measure μ on E, $(E, \mu) \in (L^*)$ implies $c(E)>0$?

Observe that the answer to this question is in the affirmative if μ is absolutely continuous with respect to the L-capacity c, or, equivalently, if c *dominates* μ (i e. $c(A) = 0$ implies $\mu(A) = 0$). An example of such a measure is the Lebesgue $2N$-dimensional measure λ_{2N} (resp. λ_N, if $E \subset \boldsymbol{R}^N$, the space of N real variables, treated as a subset of $\boldsymbol{C}^N$ consisting of those $z = (z_1, \ldots, z_N)$ that $\operatorname{Im} z_j = 0$ for $j = 1, \ldots, N$).

Remark 2.4. The fact that $\lambda_{2N}(A)>0$ (resp. $\lambda_N(A)>0$ if $A \subset \boldsymbol{R}^N$) implies $c(A)>0$ easily follows from Fubini's theorem, by means of an estimate for the extremal function associated with plane continua, and by the well-known inequality: For any compact plane set E, $4c(E) \geqslant \lambda_1(\pi_l(E))$, where $\pi_l(E)$ is the projection of E on a real line l, and λ_1 is the Lebesgue linear measure on l (see e.g. [3], relation 2.4.4). Actually, from [9], Theorem 2.3.2, we can derive much more, namely let E be a compact subset of $\boldsymbol{C}^N$ with a positive Ronkin's *Γ-capacity* $\Gamma(E)$ (see [9]). Then $\Gamma(\{z \in E\colon \Phi_E \text{ is not continuous at } z\}) = 0$. In particular, $c(\{z \in E\colon \Phi_E \text{ is continuous at } z\})>0$. We omit here the details, however.

We shall need the following lemma which is a consequence of equivalence between locally polar and globally polar sets for plurisubharmonic functions in $\boldsymbol{C}^N$, recently proved by Josefson [2].

LEMMA 2.5 ([6], Lemma 2.5). *Let E be a compact subset of $\boldsymbol{C}^N$ with $c(E)>0$. Suppose h is a holomorphic mapping in a connected open set U, $E \subset U$, with values in $\boldsymbol{C}^M$ ($M \leqslant N$), such that $c(h(U))>0$. Then $c(h(E))>0$.*

3. The main result. Given a Borel subset E of $\boldsymbol{C}^N$ and a measure μ on E let h be μ-measurable mapping defined on E, with values in $\boldsymbol{C}^M$ ($M \leqslant N$). We shall say that the pair (E, μ) *satisfies condition* (L_h^*) if for any family $\mathscr{F} \subset P(\boldsymbol{C}^M, \boldsymbol{C}^1)$ such that

$$\sup\{|f(h(z))|\colon f \in \mathscr{F}\} < \infty \qquad \mu\text{-a.e. on } E,$$

and for each $b>1$ there exist an open neighborhood G of $\bar{E}$ and a constant $M>0$ such that

$$\|f \circ h\|_G \leqslant M b^{\deg f} \qquad \text{for } f \in \mathscr{F}.$$

In the case where E is bounded, it is seen that

3.1. The pair (E, μ) satisfies (L_h^*) if and only if for each $b>1$ and for any sequence $\{f_n\} \subset P(\boldsymbol{C}^M, \boldsymbol{C}^1)$, where $\deg f_n \leqslant n$, such that

$$\sup_n \{|f_n(h(z))|\} < \infty \qquad \mu\text{-a.e. on } E$$

there exist an open neighborhood G of $\bar{E}$ and a constant $M>0$ such that for each n,

$$\|f_n \circ h\|_G \leqslant M b^n$$

(compare the proof of equivalence $(a) \Leftrightarrow$ (b) in [11]).

In the sequel we shall be interested in pairs (E, μ) which satisfy the following assumption

(Z_1) *If* $c(E)>0$, *then for every Borel subset* F *of* E *such that* $\mu(E\setminus F) = 0$, *we have* $c(F)>0$.

It is seen that if $\mu(E)>0$ and μ is *L-comparable* (i.e. either μ dominates c or c dominates μ), then (E, μ) satisfies (Z_1).

A crucial role in this paper is played by the following

LEMMA 3.2. *Let* E *be a polynomially convex, compact set in* $\mathbf{C}^N$ *and let* μ *be a measure on* E *such that the pair* (E, μ) *satisfies* (Z_1). *Suppose that* (E, μ) *satisfies also* (L^*). *Let* h *be an open holomorphic mapping defined in an open neighborhood* U *of* E, *with values in* $\mathbf{C}^M$ $(M\leqslant N)$. *Then* (E, μ) *satisfies* (L_h^*).

Proof. Take a sequence $\{f_n\}\subset P(\mathbf{C}^M, \mathbf{C}^1)$ with $\deg f_n\leqslant n$, such that

$$M(z) = \sup_n\{|f_n(h(z))|\}<\infty \quad \mu\text{-a.e. on } E,$$

and set

$$E_k = \{z\in E\colon\ M(z)\leqslant k\}.$$

Then by 2.2 and assumption (Z_1), $c(\bigcup_{k=1}^{\infty} E_k)>0$, whence by Lemma 2.1 there exists k_0 such that $c(E_k)>0$. Then by Lemma 2.5 $c(h(E_{k_0}))>0$, whence the extremal function $\Phi_{h(E_{k_0})}$ is locally bounded in $\mathbf{C}^M$. We may assume that h is bounded in U. Then, by the definition of $\Phi_{h(E_{k_0})}$, we have

$$\|f_n\circ h\|_U = \|f_n\|_{h(U)}\leqslant\|f_n\|_{h(E_{k_0})}A_1^n\leqslant k_0A_1^n$$

for $n\geqslant 1$, where $A_1 = \sup_{h(U)}\Phi_{h(E_{k_0})}(z)<\infty$. By [6], Lemma 2.1, there exist constants $A_2>0$ and $a\in(0,1)$, and polynomials $p_{n,m}\in P(\mathbf{C}^N, \mathbf{C}^1)$ with $\deg p_{n,m}\leqslant m$ such that

$$\|f_n\circ h-p_{n,m}\|_E\leqslant A_2k_0A_1^na^m, \quad n, m\geqslant 1.$$

Take a positive integer s such that $A_1a^s\leqslant a$ and set $p_n = p_{n,sn}$. Then we get

$$\|f_n\circ h-p_n\|_E\leqslant A_3a^n, \quad n\geqslant 1, \tag{3.3}$$

where $A_3 = A_2k_0$, whence

$$\sup_n\{|p_n(z)|\}<\infty \quad \mu\text{-a.e. on } E.$$

Since (E, μ) satisfies (L^*), for every $b>1$ there exist an open neighborhood G of E and a constant $A_4>0$ such that for each $n\geqslant 1$,

$$\|p_n\|_E\leqslant\|p_n\|_G\leqslant A_4B^{n/2}.$$

Hence by (3.3)

$$\|f_n\|_{h(E)} = \|f_n\circ h\|_E\leqslant A_3a^n+A_4b^{n/2}\leqslant A_5b^{n/2}$$

for $n \geqslant 1$, where $A_5 = 2\max(A_3, A_4)$. By 2.2 the set E is L-regular, whence by [6], Theorem 3.5, so is the set $h(E)$. From this it follows that there exist an open neighborhood V of $h(E)$ and a constant $A_6 > 0$ such that

$$\|f_n\|_V \leqslant A_6 b^{n/2} \|f_n\|_{h(E)} \leqslant A_7 b^n$$

for $n \geqslant 1$, with $A_7 = A_5 A_6$. Consequently, for $z \in W = h^{-1}(V \cap h(U))$ we get

$$|f_n(h(z))| \leqslant A_7 b^n, \quad n \geqslant 1 .$$

Since h is an open mapping, the set W is an open neighborhood of E, and this completes the proof of the lemma.

Now we can prove the main result of this paper.

THEOREM 3.4. *Suppose that E is a polynomially convex compact set in $\mathbf{C}^N$ and h is an open holomorphic mapping defined in an open neighborhood U of E, with values in $\mathbf{C}^M$ $(M \leqslant N)$. Let μ and ν be measures on E and $h(E)$, respectively, such that the triplet (μ, h, ν) satisfies the assumption*

(Z_2) *For each Borel subset F of $h(E)$, $\nu(F) = 0$ implies $\mu(h^{-1}(F) \cap E) = 0$.*
Then, if the pair (E, μ) satisfies both (Z_1) and (L^*) so does the pair $(h(E), \nu)$.

Proof. Take a sequence $\{f_n\} \subset P(\mathbf{C}^M, \mathbf{C}^1)$, where $\deg f_n \leqslant n$, $n \geqslant 1$, and suppose that

$$\sup_n \{|f_n(w)|\} < \infty \quad \nu\text{-a.e. on } h(E) .$$

Then by assumption (Z_2)

$$\sup_n \{|f_n(h(z))|\} < \infty \quad \mu\text{-a.e. on } E .$$

By Lemma 3.2, for every $b > 1$ there exist an open neighborhood G of E and a constant $A > 0$ such that

$$\|f_n\|_{h(G)} = \|f_n \circ h\|_G \leqslant A b^n, \quad n \geqslant 1 .$$

Since h is an open mapping, the set $h(G)$ is open, and, consequently, $(h(E), \nu)$ satisfies (L^*) It easily follows from assumption (Z_2) and from Lemma 2.5 that $(h(E), \nu)$ satisfies also (Z_1). The theorem is proved.

4. Generalizations and special cases.

Remark 4.1. The assumption of Theorem 3.4 that h is open is too strong. For instance, it suffices to assume that the pair (E, h) satisfies the following requirement

(H) *For each bounded open set V such that $E \subset V \subset \overline{V} \subset U$, the extremal function $\Phi_{h(\overline{V})}$ is continuous at every point of the set $h(E)$.*

Then by Lemma 2.5 and [6], theorem 3.5, our Lemma 3.2 and, consequently, Theorem 3.5 remain to hold.

An example of a pair (E, h) satisfying (H), where h is not open, is given by $E = [0,1]\times[0,1]\subset \boldsymbol{R}^2$ and $h(z_1, z_2) = (z_1, z_2 g(z_1))$, where g is an analytic function on $[0,1]$ with $g(0) = 0$ and $g(t)>0$ for $t\in(0,1]$ (see [7], example 3.1).

Remark 4.2. It is well-known that every non-constant holomorphic function $h\colon \boldsymbol{C}^N\supset U\to\boldsymbol{C}^1$, where U is connected, is an open mapping. Therefore, for $M = 1$, we obtain a very nice corollary to Theorem 3.2.

Suppose now that $\mu = \lambda_{2N}$, the $2N$-dimensional Lebesgue measure, and E is a compact subset of $\boldsymbol{C}^N$ with $\lambda_{2N}(E)>0$. Then by Remark 2.4 the pair (E, λ_{2N}) satisfies assumption (Z_1). Let h be a holomorphic mapping in an open neighborhood U of E, with values in $\boldsymbol{C}^M$ $(M\leqslant N)$. Denote by K the set of all critical points of h,

$$K = \{z\in U\colon \operatorname{rank}_z h<M\}.$$

We have

PROPOSITION 4.3. *Suppose that K does not contain any connected component of U. Then the triplet $(\lambda_{2N}, h, \lambda_{2M})$ satisfies (Z_2).*

Proof. Let F be a Borel subset of the set $h(E)$ such that $\lambda_{2M}(F) = 0$. Set $I = h^{-1}(F)\cap E$. We have to prove that $\lambda_{2N}(I) = 0$. Suppose $\lambda_{2N}(I)>0$. The assumption on K means that K is an analytic subset of U, whence K is globally $\boldsymbol{C}^N$-polar (see [2]), and consequently, by [10], Theorem 3.10 and Corollary 3.9, we have $c(K) = 0$. Therefore by Remark 2.4, $\lambda_{2N}(K) = 0$. Consequently, $\lambda_{2N}(I\setminus K)>0$, and since the space $\boldsymbol{C}^N$ is a Lindelöf space, there exists a point $a\in I\setminus K$ such that for each open neighborhood V of a, $\lambda_{2N}((I\setminus K)\cap V)>0$. Since K is a closed subset of U, we can choose V to be disjoint with K. Since $\operatorname{rank}_a h = M$, there exists a nonsingular affine mapping l in $\boldsymbol{C}^N$ such that $l(0) = a$ and

$$\det\left[\frac{\partial(h_i\circ l)}{\partial z_j}(0)\right]\neq 0,$$

where $i, j = 1, \ldots, M$. Hence we can find a neighborhood $W\subset l^{-1}(V)$ such that the mapping

$$\tilde{h}\colon z = (z_1, \ldots, z_N)\to\big(h_1(l(z)), \ldots, h_M(l(z)), z_{M+1}, \ldots, z_N\big)$$

is a biholomorphism of W onto $\tilde{h}(W)$. Therefore

$$\lambda_{2N}\big(\tilde{h}(l^{-1}(I\setminus K)\cap W)\big) = \int_{l^{-1}(I\setminus K)\cap W} |J_{\tilde{h}}(z)|^2\, d\lambda_{2N}>0,$$

where $J_{\tilde{h}}$ denotes the determinant of the jacobian matrix of $\tilde{h}$, whence $\lambda_2\big(\tilde{h}(l^{-1}(I))\big)>0$. But

$$\tilde{h}(l^{-1}(I))\subset F\times\pi_{N-M}(l^{-1}(I)),$$

where

$$\pi_{N-M}\colon \boldsymbol{C}^N\ni(w_1, \ldots, w_N)\to(w_{M+1}, \ldots, w_N)\in\boldsymbol{C}^{N-M},$$

and therefore $\lambda_{2M}(F)>0$, a contradiction. Consequently, $\lambda_{2N}(I) = 0$ as asserted.

Remark 4.4. If $E \subset \boldsymbol{R}^N$, and $h(E) \subset \boldsymbol{R}^M$, then Proposition 4.3 remains to hold if we replace λ_{2N} and λ_{2M} by λ_N and λ_M, respectively. This can be proved by a similar argument to that of the proof of Proposition 4.3.

As a consequence of Theorem 3.4, Remark 4.1, Proposition 4.3, and Remark 4.4, we obtain

COROLLARY 4.5. *Let E be a polynomially convex, compact set in $\boldsymbol{C}^N$ (resp. any compact set in $\boldsymbol{R}^N$). Let h be a holomorphic mapping in an open neighborhood U of E, with values in $\boldsymbol{C}^M$ ($M \leqslant N$), such that*

$$\operatorname{int}\{z \in U: \operatorname{rank}_z h < M\} = \emptyset .$$

If then (E, λ_{2N}) (resp. (E, λ_N)) satisfies (L^*) and (E, h) satisfies assumption (H) *(and $h(E) \subset \boldsymbol{R}^M$, if $E \subset \boldsymbol{R}^N$), then the pair $(h(E), \lambda_{2M})$ (resp. $(h(E), \lambda_M)$) satisfies (L^*).*

5. Condition (L_0^*). Given a Borel subset E of $\boldsymbol{C}^N$ and a measure μ on E, the pair (E, μ) is said *to satisfy condition* (L_0^*) if for every family $\mathscr{F} \subset P(\boldsymbol{C}^N, \boldsymbol{C}^1)$ such that

$$\sup_{f \in \mathscr{F}} |f(z)| < \infty \qquad \mu\text{-a.e. on } E,$$

and for every $b > 1$ there exists a constant $A > 0$ such that

$$\|f\|_E \leqslant A b^{\deg f}, \quad f \in \mathscr{F} .$$

It follows from the definition and from 1.2 that

5.1. The pair (E, μ) satisfies (L^*) if and only if it satisfies both (L_0^*) and (B_0).

The condition (L_0^*) is, in general, essentially weaker than (L^*). Indeed, by taking $E = [0, 1] \cup \{2\} \subset \boldsymbol{R}^1$ and $\mu(A) = \lambda_1(A)$, if $2 \notin A$, and $\mu(A) = \lambda_1(A) + 1$, if $2 \in A$, where A is a Borel subset of E, we obtain a pair which does not satisfy (L^*) (the extremal function Φ_E is not continuous at 2!), which by Example 1.1, however, does satisfy (L_0^*).

Observe also that for the Cantor ternary set $E \subset [0, 1]$ we have $\lambda_1(E) = 0$, and therefore the pair (E, λ_1) does not satisfy (L^*). E is, however, known to be L-regular.

An inspection of the proofs of Lemma 3.2 and Theorem 3.4 gives the following

THEOREM 5.2. *Let h be an open holomorphic mapping defined in an open neighborhood U of a polynomially convex, compact set E in $\boldsymbol{C}^N$, with values in $\boldsymbol{C}^M$ ($M \leqslant N$). Suppose μ and ν are measures on E and $h(E)$, respectively, such that the triplet (μ, h, ν) satisfies assumption* (Z_2). *Then, if the pair (E, μ) satisfies both* (Z_1) *and* (L_0^*) *so does the pair $(h(E), \nu)$.*

Remark 5.3. The assumption of Theorem 5.2 that h is open may be replaced by the weaker one that for any subset F of E with $c(F) > 0$ we have $c(h(F)) > 0$. By Lemma 2.5, the last assumption is equivalent to the following one:

For each connected component V of U with $c(E \cap V) > 0$, we have $c(h(V)) > 0$.

Similarly as Corollary 4.5, by Proposition 4.3, Remark 4.4, Theorem 5.2, and Remark 5.3, we get

COROLLARY 5.4. *Let E be a polynomially convex, compact set in $\mathbf{C}^N$ (resp. any compact set in $\mathbf{R}^N$). If the pair (E, λ_{2N}) (resp. (E, λ_N)) satisfies (L_0^*) and h satisfies the assumptions of Remark 5.3 (and $h(E)\subset\mathbf{R}^M$ if $E\subset\mathbf{R}^N$), then the pair $(h(E), \lambda_{2M})$ (resp. $(h(E), \lambda_M)$) also satisfies (L_0^*).*

In fact, to conclude the corollary we have to show that with the assumptions of Remark 4.3 on h we can apply Proposition 4.3. This, however, follows from

LEMMA 5.5. *Let h be a holomorphic mapping in a non-void connected open subset U of $\mathbf{C}^N$, with values in $\mathbf{C}^M$ $(M\leqslant N)$, such that $c(h(U))>0$. Then*

$$c(\{z\in U\colon \operatorname{rank}_z h<M\}) = 0 .$$

Proof. It follows from the assumptions on h that for any Borel subset E of U, $c(E)>0$ implies $c(h(E))>0$ (compare the proof of Lemma 2.5 in [6]). Since the set

$$K = \{z\in U\colon \operatorname{rank}_z h<M\}$$

is an analytic subset of U, we have $c(K) = 0$ unless that $K = U$ (see [2] and [10], Theorem 3.10 and Corollary 3.9). Suppose then $K = U$ and put $k = \max\{\operatorname{rank}_z h\colon z\in U\}$. The set $A_k = \{z\in U\colon \operatorname{rank}_z h<k\}$ is then analytic, whence $c(U\setminus A_k)>0$. Moreover, for each point $a\in U\setminus A_k$, there exists a neighborhood V of a such that $h(V)$ is a k-dimensional complex submanifold of $\mathbf{C}^M$ (see e.g. [1], Theorem VB 10), and therefore $h(U\setminus A_k)$ is locally polar (for plurisubharmonic functions in $\mathbf{C}^M$). Hence by [10], Theorem 3.10 and Corollary 3.9, $c(h(U\setminus A_k)) = 0$, a contradiction.

6. New examples of pairs (E, μ) satisfying condition (L^*). Given a Borel subset E of $\mathbf{C}^N$ and a measure μ on E, we say that the pair (E, μ) *satisfies condition (L^*) at a point $a\in\mathbf{C}^N$* if for every family $\mathscr{F}\subset P(\mathbf{C}^N, \mathbf{C}^1)$ such that

$$\sup\{|f(z)|\colon f\in\mathscr{F}\}<\infty \quad \mu\text{-a.e. on } E,$$

and for every $b>1$ there exist a neighborhood V of a and a constant $A>0$ such that for each $f\in\mathscr{F}$,

$$\|f\|_V\leqslant Ab^{\deg f} .$$

It is obvious that

6.1. If E is compact, then (E, μ) satisfies (L^*) if and only if (E, μ) satisfies (L^*) at every point $a\in E$.

Note the following sufficient condition for a pair (E, μ) to satisfy condition (L^*) at a point.

PROPOSITION 6.2. *Let E be a Borel subset of $\mathbf{C}^N$ and let μ be a measure on E. Given $a\in\mathbf{C}^N$, suppose that for every $b>1$ there exists a compact subset F of E such that*

1° the pair (F, μ) satisfies (L_0^*),

1° the extremal function Φ_F is continuous at a and $\Phi_F(a)<b$.

Then the pair (E, μ) satisfies (L^*) at a.

Specially convenient is the following criterion of the L^*-regularity (with respect to the Lebesgue measure) which is an analogue of a criterion for the L-regularity due to Gončar (see [7]).

ATTAINMENT CRITERION 6.3. *Suppose that for a Borel subset E of $\mathbf{C}^N$ (resp. $\mathbf{R}^N$) and a point $a \in \mathbf{C}^N$ there exists a real line l and a sequence $\{K_n\}$ of compact subsets of l such that*

1° *for each n, (K_n, λ_1) satisfies (L^*) (in l),*

2° $K_n \subset \operatorname{int} E$ *(resp. $K_n \subset \operatorname{int}_{\mathbf{R}^N} E$), $n = 1, 2, \ldots$*

3° $\Phi_{K_n}(a) \to 1$, *as $n \to \infty$.*

Then the pair (E, λ_{2N}) (resp. (E, λ_N)) satisfies condition (L^*) at a.

Proof. Consider the case where $E \subset \mathbf{R}^N$. Let l be the line chosen to a. We may assume that $a = 0 \in \mathbf{R}^N$ and $l = \{(t, 0, \ldots, 0) \in \mathbf{R}^N \colon t \in \mathbf{R}^1\}$. Denote by π the projection $\pi\colon \mathbf{R}^N \ni x = (x_1, \ldots, x_N) \to x_1 \in \mathbf{R}^1$.

By 1° the pair $(\pi(K_n), \lambda_1)$ satisfies (L^*) in $\mathbf{R}^1$. Fix $b > 1$. By 3° there exists n such that

$$\Phi_{\pi(K_n)}(0) < b\,.$$

By 2° we can find $\varepsilon > 0$ such that the set

$$F_n(\varepsilon) = \{x \in \mathbf{R}^n \colon x_1 \in \pi(K_n),\ |x_j| \leqslant \varepsilon,\ j = 2, \ldots, N\}$$

is contained in the interior of E. By Example 1.1, 1° and by Fubini's theorem the pair $(F_n(\varepsilon), \lambda_N)$ satisfies (L^*). Moreover, by [10], Proposition 5.9,

$$\Phi_{F_n(\varepsilon)}(x) = \max\{\Phi_{\pi(K_n)}(x_1),\ \Phi_{P(\varepsilon)}(x_2, \ldots, x_N)\}\,,$$

for $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$, where $P(\varepsilon)$ is the parallelepiped $\{|x_j| \leqslant \varepsilon,\ j = 2, \ldots, N\} \subset \mathbf{R}^{N-1}$, whence

$$\Phi_{F_n(\varepsilon)}(0) = \Phi_{\pi(K_n)}(0) < b\,.$$

Thus, by Proposition 6.2, (E, λ_N) satisfies (L^*) at 0.

A similar proof works in the case where $E \subset \mathbf{C}^N$.

Now we can give some new examples of pairs (E, μ) satisfying (L^*).

Example 6.4. Let $G = \{(x, y) \in \mathbf{R}^2 \colon f(x) < y < g(x),\ 0 < x < 1\}$, where f is an analytic function in a neighborhood of $[0, 1]$, g is continuous on $[0, 1]$, and $f(x) < g(x)$ for $x \in (0, 1]$. Set $E = \bar{G}$. Then the pair (E, λ_2) satisfies (L^*).

Proof. We may assume that $f(x) \equiv 0$. (Indeed, take the biholomorphism $h(x, y) = (x, y - f(x))$ of a neighborhood (in $\mathbf{C}^2$) of E into $\mathbf{C}^2$. Then by Corollary 4.5 the pair (E, λ_2) satisfies (L^*) if and only if so does the pair $(h(E), \lambda_2)$.) By 6.1 it suffices to prove that (E, λ_2) satisfies (L^*) at every point $a \in E$. If $a \neq (0, 0)$, this easily follows from Attainment Criterion 6.3. Suppose then $a = (0, 0)$. Given $b > 1$, choose $\alpha \in (0, 1)$ so close to 0 that

$$\Phi_{[\alpha, 1]}(0) < b\,.$$