

## On the distribution of zeros of the polynomials of best $L^2$ -approximation to holomorphic functions

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**0. Introduction.** It is known ([1], p. 450) that if the polynomials of best approximation to a continuous function  $f$  on  $[-1, 1]$ , in the sense of the norm of the space  $L^2([-1, 1], \omega dx)$  with a special positive continuous weight  $\omega$  on  $[-1, 1]$ , have no zeros in an open neighborhood (in the complex plane  $\mathbb{C}$ ) of  $[-1, 1]$ , then  $f$  has an analytic extension to a neighborhood of  $[-1, 1]$ . In this paper this effect is proved to hold for  $f \in L^2(E, \mu)$ , where  $E$  is any  $L$ -regular compact subset of  $\mathbb{C}$  and  $\mu$  is a finite positive measure on  $E$  such that the pair  $(E, \mu)$  satisfies a certain polynomial condition. Actually we prove more precise results (Theorems 4.1 and 4.8, and Proposition 4.9) Our main tool is Leja's extremal function associated with a compact set in  $\mathbb{C}$ .

**1. Extremal function.** Let  $E$  be a compact set in the complex plane  $\mathbb{C}$  with a positive transfinite diameter  $d(E)$ . It is known [3] that there exists a sequence  $\{z_m\}_{m=0}^{\infty}$  of distinct points of the boundary  $\partial E$  of the set  $E$ , called Leja's *extremal sequence* associated with  $E$ , such that

$$|l_m(z)| \leq |l_m(z_m)|$$

for each  $z \in E$  and  $m = 1, 2, \dots$ , and

$$(1.1) \quad d(E) = \lim_{m \rightarrow \infty} |l_m(z_m)|^{1/m},$$

where

$$(1.2) \quad l_m(z) = (z - z_0)(z - z_1) \dots (z - z_{m-1})$$

for  $m = 1, 2, \dots$ . Moreover, the sequence  $\{|l_m(z)/l_m(z_m)|^{1/m}\}$  is uniformly convergent on every compact subset of  $D_{\infty}(E)$  to Leja's *extremal function*  $L_E$  of  $E$ , which is known to be equal to

$$(1.3) \quad L_E(z) = \begin{cases} 1, & z \in \mathbb{C} \setminus D_{\infty}(E), \\ \exp G(z), & z \in D_{\infty}(E), \end{cases}$$

where  $D_{\infty}(E)$  denotes the unbounded connected component of the set  $\mathbb{C} \setminus E$  and  $G$  is the generalized Green function (with pole at  $\infty$ ) of the set  $D_{\infty}(E)$ . It is seen that the only

points of discontinuity of the function  $L_E$  are those points of the boundary  $\partial D_\infty(E)$  of  $D_\infty(E)$  which are not regular with respect to the Green function  $G$ . In particular, if for each point  $a \in \partial D_\infty(E)$ ,

$$\lim_{D_\infty(E) \ni z \rightarrow a} G(z) = 0,$$

then the function  $L_E$  is continuous in  $C$ . In such a case the set  $E$  is called *L-regular*. Then for every  $r > 1$  the set

$$E_r = \{z \in C: L_E(z) < r\}$$

is an open neighborhood of  $E$ . Moreover, if  $1 < r_1 < r$ , then the closure  $\bar{E}_{r_1}$  of  $E_{r_1}$  is contained in  $E_r$ .

From relations (1.1)–(1.3) we derive the following lemma which will play a crucial role in this note.

LEMMA 1.4. *Suppose  $d(E) > 0$  and  $a \neq 0$ . If all zeros of the polynomial*

$$p(z) = a(z - c_1) \dots (z - c_n)$$

are in  $D_\infty(E)$ , then

$$|a| [d(E)]^n L_E(c_1) \dots L_E(c_n) = M,$$

where

$$M = \lim_{m \rightarrow \infty} |p(z_0)p(z_1) \dots p(z_{m-1})|^{1/m},$$

$\{z_m\}$  being Leja's extremal sequence associated with  $E$ .

We shall also need the well-known Bernstein–Walsh inequality (see [7], Lemma 2.1)

1.5. For any polynomial  $p$ ,

$$|p(z)| \leq \|p\|_E [L_E(z)]^{\deg p}, \quad z \in C,$$

where  $\|p\|_E = \sup_{z \in E} |p(z)|$  and  $\deg p$  denotes the degree of  $p$ .

**2. Space  $L_p(E, \mu)$ .** Let  $E$  be an infinite compact set in  $C$  and let  $\mu$  be a positive finite measure defined on  $E$ . Denote by  $P_n$  the set of all polynomials from  $C$  to  $C$  of degree  $\leq n$  and set  $P = \bigcup_{n=0}^{\infty} P_n$ . Let  $L_p^2(E, \mu)$  be the closure of  $P$  in the Hilbert space  $L^2(E, \mu)$ .

Put  $w_0 \equiv 1$  and denote by  $w_n$  ( $n = 1, 2, \dots$ ) a polynomial of the form  $z^n + u$ , where  $u \in P_{n-1}$ , such that

$$(2.1) \quad \|w_n\| = \inf\{\|z^n + v\|: v \in P_{n-1}\},$$

$\|\cdot\|$  denoting the norm in  $L^2(E, \mu)$ . Since  $P_{n-1}$  is a finite-dimensional subspace of  $L^2(E, \mu)$  there is exactly one polynomial  $w_n$  satisfying (2.1). Moreover, the sequence  $\{w_n\}$

is orthogonal, since, otherwise, for  $(w_n, w_m) = a \neq 0$  with  $m < n$  we would have  $\|w_n - (a/\|w_m\|^2)w_m\| < \|w_n\|$ , a contradiction. Therefore the polynomials

$$(2.2) \quad e_n = w_n/\|w_n\|, \quad n = 0, 1, \dots$$

form an orthonormal basis in the Hilbert space  $L_p(E, \mu)$ .

**3. Conditions  $(L^*)$  and  $(L_0^*)$ .** Let  $E$  be a Borel subset of the space  $C$  and let  $\mu$  be a positive finite measure on  $E$ . The pair  $(E, \mu)$  is said to satisfy condition  $(L^*)$  if for every family  $\mathcal{F} \subset P$  such that

$$(3.1) \quad \sup_{f \in \mathcal{F}} |f(z)| < \infty \quad \mu\text{-almost everywhere on } E,$$

and every  $b > 1$  there exist an open neighborhood  $G$  of  $\bar{E}$ , the closure of  $E$ , and a positive constant  $M$  such that for each  $f \in \mathcal{F}$  we have

$$\sup_{z \in G} |f(z)| \leq Mb^{\deg f}.$$

The pair  $(E, \mu)$  is said to satisfy condition  $(L_0^*)$ , if for every family  $\mathcal{F} \subset P$  satisfying (3.1) and every  $n > 1$  there exists a constant  $M > 0$  such that for each  $f \in \mathcal{F}$ ,

$$\|f\|_E \leq Mb^{\deg f}.$$

It follows from the definition that

3.2. If  $\mu$  and  $\nu$  are positive finite measures on a Borel subset  $E$  of  $C$  such that  $\nu$  dominates  $\mu$  (i.e.  $\nu(A) = 0$  implies  $\mu(A) = 0$  for every Borel subset  $A$  of  $E$ ), and the pair  $(E, \mu)$  satisfies  $(L^*)$  (resp.  $(L_0^*)$ ), then so does the pair  $(E, \nu)$ .

It can be proved (see [6]) that

3.3. If  $E$  is compact, then the pair  $(E, \mu)$  satisfies  $(L^*)$  if and only if  $E$  is  $L$ -regular and  $(E, \mu)$  satisfies  $(L_0^*)$ .

The first example of a pair  $(E, \mu)$  satisfying  $(L^*)$  comes from Leja [2] and is the following

Example 3.4. Let  $E$  be a Jordan rectifiable arc in  $C$  and let  $\mu$  be the length measure on  $E$ . Then  $(E, \mu)$  satisfies  $(L^*)$ . In particular,  $E$  can be thought of as a compact interval in  $R$ , the space of reals, and  $\mu = \lambda$ , the linear Lebesgue measure.

Nguyen Thanh Van [4] proved that

3.5. For any compact set  $E$  in  $C$  and any positive finite measure  $\mu$  on  $E$ , the pair  $(E, \mu)$  satisfies  $(L_0^*)$  if and only if

$$\limsup_{n \rightarrow \infty} (\|e_n\|_E)^{1/n} \leq 1,$$

where  $\{e_n\}$  is the sequence of the polynomials defined by (2.2).

On the other hand, Widom [9] showed that if  $E$  is  $L$ -regular and  $\mu$  is an admissible (in the sense of [9]) measure on  $E$ , then

$$\|e_n\|_E = [1 + o(1)]^n, \quad \text{as } n \rightarrow \infty.$$

Hence we get

Example 3.6 ([4]). If  $E$  is an  $L$ -regular compact subset of  $C$  and  $\mu$  is an admissible (in the sense of [9]) measure on  $E$ , then  $(E, \mu)$  satisfies both conditions  $(L^*)$  and  $(L_0^*)$ . In particular  $\mu$  can be thought of as the harmonic measure on  $E$ .

Other examples of pairs  $(E, \mu)$  satisfying  $(L^*)$  (resp.  $(L_0^*)$ ) can be derived from the main theorem of [6], which, roughly speaking, states the invariance of the conditions  $(L^*)$  and  $(L_0^*)$  under non-constant holomorphic mappings defined in an open neighborhood of  $E$  (in [6] the multidimensional case is also treated).

We note a lemma which easily follows from 3.5.

LEMMA 3.7 ([4]). *If  $(E, \mu)$  satisfies  $(L_0^*)$ , then for every  $b > 1$  there is a constant  $M > 0$  such that*

$$\|f\|_E \leq Mb^{\deg f} \|f\|$$

for all polynomials  $f \in P$ .

**4. Main result.** Let  $E$  be a compact set in  $C$  and let  $\mu$  be a positive finite measure on  $E$ . Given a function  $f \in L^2(E, \mu)$  we consider the Fourier series of  $f$  with respect to the basis (2.2),  $\sum_{k=0}^{\infty} a_k e_k$ , where  $a_k = (f, e_k)$ . Put  $f_n = \sum_{k=0}^n a_k e_k$ . Assume that  $f$  is not equal to a constant  $\mu$ -a.e. on  $E$ . We shall prove the following

THEOREM 4.1. *Let  $r > 1$ . If the pair  $(E, \mu)$  satisfies  $(L^*)$ , then the following conditions are equivalent*

(i) *for any number  $r_1 \in (1, r)$  there exists a complex number  $a$  such that  $f_n(z) + a \neq 0$  for  $z \in E_{r_1} = \{z \in C: L_E(z) < r_1\}$ , and  $n = 0, 1, \dots$ ;*

(ii) *for any number  $r_1 \in (1, r)$  there exists a constant  $a \in C$  such that*

$$f_n(z) + a = b_n(z - c_{n,1}) \dots (z - c_{n,s_n}) \neq 0$$

for  $z \in C \setminus D_{\infty}(E)$ , where  $s_n = \deg f_n$ , and

$$\liminf_{n \rightarrow \infty} [L_E(c_{n,1}) \dots L_E(c_{n,s_n})]^{1/s_n} \geq r_1;$$

(iii)  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq r^{-1}$ ;

(iv) *there exists a holomorphic function  $\tilde{f}$  in  $E_r$  such that  $\tilde{f}(z) = f(z)$   $\mu$ -a.e. on  $E$ .*

Proof. It is evident that (i) implies (ii). Suppose now (ii). By the Bessel inequality we have

$$\|f_n\| \leq \|f\|, \quad n = 0, 1, \dots$$

Hence by Lemma 3.7 for each number  $b > 1$  there is a constant  $M > 0$  such that

$$(4.2) \quad \|f_n\|_E + |a| \leq Mb^n, \quad n = 0, 1, \dots,$$

where  $a$  is chosen to  $r_1 \in (1, r)$ . Observe now that if  $a_n \neq 0$ , that is if  $s_n = n$ , then  $a_n = b_n \|w_n\|$ . Hence by (4.2) and Lemma 1.4,

$$(4.3) \quad |a_n| \leq M [b/d(E)]^n [L_E(c_{n+1}) \dots L_E(c_{nn})]^{-1} \|w_n\|.$$

Let  $t_n$  denotes the  $n$ -th Čebyšev polynomial associated with the compact  $E$ . Then we have

$$\|t_n\|_E = \inf\{\|z^n + v\|_E : v \in P_{n-1}\}.$$

It is known (see e.g. [8]) that

$$\lim_{n \rightarrow \infty} \|t_n\|_E^{1/n} = d(E).$$

Hence

$$(4.4) \quad \|w_n\| \leq [\mu(E)]^{1/2} \|t_n\|_E \leq [\mu(E)]^{1/2} [d(E) + o(1)]^n,$$

whence by (4.3),

$$|a_n| \leq N [b/r_1 + o(1)]^n,$$

where  $N = M[\mu(E)]^{1/2}$ . Thus, by the arbitrariness of the choice of  $b$  and  $r_1$ , we get (iii).

(iii)  $\Rightarrow$  (iv). It suffices to show that the series  $\sum a_n e_n$  is uniformly convergent on every compact subset of  $E_r$ . With this aim choose numbers  $r_i$  ( $i = 1, 2, 3$ ) so that  $1 < r_1 < r_2 < r_3 < r$ . By (iii), for  $n \geq n_0 = n_0(r_3)$  we have

$$(4.5) \quad |a_n| \leq r_3^{-n}.$$

Since  $\|e_n\| = 1$ , by Lemma 3.7 there exists a constant  $M > 0$  such that

$$\|e_n\|_E \leq M (r_3/r_2)^n, \quad n = 0, 1, \dots$$

Hence by (4.5) and (1.5)

$$\|a_n e_n\|_{E_{r_1}} \leq M (r_1/r_2)^n,$$

whence the series  $\sum a_n e_n$  is normally convergent in  $E_{r_1}$ . Since  $E_r = \bigcup_{1 < r_1 < r} E_{r_1}$ , by the arbitrariness of the choice of  $r_1 \in (1, r)$  we get the result.

(iv)  $\Rightarrow$  (i). By the Bernstein-Walsh theorem (see [8]) there exists a sequence of polynomials  $p_n \in P_n$  ( $n = 1, 2, \dots$ ) uniformly convergent to  $\tilde{f}$  on every compact subset of  $E_r$  and such that

$$\limsup_{n \rightarrow \infty} (\|\tilde{f} - p_n\|_E)^{1/n} \leq r^{-1}.$$

Hence, since for each  $n > 1$ ,  $\|f - f_n\| \leq \|f - p_n\| = \|\tilde{f} - p_n\|$ , for every  $r_1 \in (1, r)$  we have

$$\|f_n - p_n\| \leq r_1^{-n}$$

for  $n \geq n_0 = n_0(r_1)$ . Thus applying Lemma 3.7 and inequality 1.5 we show that the sequence  $\{f_n\}$  is also uniformly convergent on every compact subset of  $E_r$ , whence we get (i) (for each  $r_1 \in (1, r)$ , take  $a \in \mathbb{C}$  such that  $|a| > \sup\{\|f_n\|_{E_{r_1}} : n = 0, 1, \dots\}$ ).

Remark 4.6. If  $r = \infty$ , then each of conditions (i)–(iii) is equivalent to the entirety of the function  $\tilde{f}$  of (iv).

Remark 4.7. If  $E = [-1, 1]$  and  $d\mu = q(x)(1-x)^\alpha(1+x)^\beta d\lambda$ , where  $q$  is a positive continuous function defined on  $E$ ,  $\alpha$  and  $\beta$  are reals and  $\lambda$  is the Lebesgue linear measure, then the equivalence (i) $\Leftrightarrow$ (iv) was proved by Bernstein [1], p. 450.

The equivalence (iii) $\Leftrightarrow$ (iv) is well-known under more restrictive assumptions on  $E$  and  $\mu$  (see [8]).

As a corollary to Theorem 4.1 we obtain its sharper version as follows.

**THEOREM 4.8.** *With the notations and assumptions of Theorem 4.1, the following conditions are equivalent*

(i) *for any number  $r_1 \in (1, r)$  there exists  $a \in C$  such that  $f_n(z) + a \neq 0$  for  $z \in E_{r_1}$ ,  $n = 0, 1, \dots$ , and for each  $s > r$ ,*

$$\bigcup_{n=0}^{\infty} f_n(E_s) = C;$$

(ii) *for any number  $r_1 \in (1, r)$  there exists  $a \in C$  such that*

$$f_n(z) + a = b_n(z - c_{n,1}) \dots (z - c_{n,s_n}) \neq 0$$

*for  $z \in C \setminus D_\infty(E)$  and*

$$\liminf_{n \rightarrow \infty} [L_E(c_{n,1}) \dots L_E(c_{n,s_n})]^{1/s_n} = r_1;$$

(iii)  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} = r^{-1}$ ;

(iv) *there exists a holomorphic function  $\tilde{f}$  in  $E_r$  having at least one singular point on the boundary  $\partial E_r$  of  $E_r$  and such that  $\tilde{f}(z) = f(z)$   $\mu$ -a.e. on  $E$ ,*

If the polynomials  $f_n$  of (i), Theorem 4.1 (or Theorem 4.8) are not polynomials of best approximation to  $f$ , then implication (i) $\Rightarrow$ (iv) is, in general, false, namely given a positive continuous function  $f$  on  $E = [-1, 1]$  it is not difficult to construct a sequence  $\{p_n\}$  of polynomials tending uniformly to  $f$  on  $[-1, 1]$  such that for each  $r > 1$ ,  $p_n(z) \neq 0$  for  $z \in E_r$ ,  $n \geq n_0 = n_0(r)$  (see [1], p. 449). However, a small change of the polynomials  $f_n$  is possible leaving implication (i) $\Rightarrow$ (iv) to hold true. We have

**PROPOSITION 4.9.** *Let  $(E, \mu)$  satisfy  $(L^*)$ . Suppose  $\{p_n\}$  is a sequence of polynomials,  $\deg p_n \leq n$ ,  $n = 1, 2, \dots$  satisfying the following requirements*

1° *there exists an open neighborhood  $U$  of  $E$  such that  $p_n(z) \neq 0$ ,  $z \in U$ ,  $n = 1, 2, \dots$ ;*

2°  $\inf_n |p_n(z)| = m(z) > 0$ ,  $z \in E$ ;

3°  $\limsup_{n \rightarrow \infty} \|f_n - p_n\|^{1/n} = a < 1$ .

*Then there exists a holomorphic function  $\tilde{f}$  in an open neighborhood of  $E$  such that  $\tilde{f}(z) = f(z)$   $\mu$ -a.e. on  $E$ .*

**Proof.** Take  $b \in (a, 1)$ . Since  $(E, \mu) \in (L^*)$ , by 3° and Lemma 3.7 and 1.5, there exist a constant  $M > 0$  and an open neighborhood  $U$  of  $E$  such that

$$\sup_{z \in U} |f_n(z) - p_n(z)| \leq Mb^n, \quad n = 1, 2, \dots$$

Choose now  $c \in (b, 1)$ . By 1°, 2° and [5], Lemma 3.2, there exist a constant  $A > 0$  and an open neighborhood  $V$  of  $E$ ,  $V \subset U$ , such that

$$|p_n(z)| \geq Ac^n, \quad z \in V.$$

Hence

$$|f_n(z)| \geq |p_n(z)| - |f_n(z) - p_n(z)| \geq Ac^n - Mb^n$$

for  $z \in V$ ,  $n = 1, 2, \dots$ , whence  $f_n(z) \neq 0$  for  $z \in V$  and  $n$  sufficiently large. Therefore by Theorem 4.1 there exist an open neighborhood  $W$  of  $E$ ,  $W \subset V$ , and a holomorphic function  $\tilde{f}$  in  $W$  such that  $\tilde{f}(z) = f(z)$   $\mu$ -a.e. on  $E$ . Obviously, the polynomials  $p_n$  tend to  $\tilde{f}$  uniformly in an open neighborhood of  $E$ .

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