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The convergence of generalized sequence of sets and functions in locally convex spaces, I.

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1. Introduction

Let F and G be two linear spaces over the reals $\boldsymbol{R}$ paired by a bilinear form $\langle .,. \rangle$: $F \times G \to \boldsymbol{R}$, which separates points both in F and in G. Let $\Gamma(F)$ and $\mathscr{K}(F)$ denote, respectively, the family of all regular functions defined on F and the family of sets in Carthesian product $F \times \boldsymbol{R}$ which are epigraphs of regular functions on F (for precise definitions see Section 2 below).

In this paper we give an unified approach to various modes of convergence ((K)-, (Q)-, (o)-convergence) of generalized sequences (i.e. nets) $\{K_\iota\}_J \subset \mathscr{K}(F)$ or $\{f_\iota\}_J \subset \Gamma(F)$, where J denotes a directed set. We also give some new results concerning the relations between these notions and between the accompaning notions like semicontinuity, continuity, property (K) or (Q) of a net (for the definitions see Section 3).

The notions mentioned above generalize or are connected with some notions well known in the literature. For instance, the (Q)- or (o)-convergences generalize, respectively, the G- and C-convergences defined by Marcellini [15] in the case of a denumerable sequence of functions on a reflexive and separable Banach space; compare also Goodman [10]. Nevertheless, the original definition of G-convergence, formulated for sequences of parabolic or elliptic operators in divergence form, goes back to Spagnolo [24], see also de Giorgi and Spagnolo [9] and references in these papers. It should be emphasized that many interesting problems in Mathematical Physics concerning the heat conduction, electro- and magnetostatics may be formulated in terms of G-convergence; see for instance Sanchez Palencia [23] § 2.1. The (o)-convergence is also connected with the convergence in the sense of Mosco [17], [18] and its extension given by Joly [11]. For the (K)-convergence we refer to Kuratowski [14] and Frolik [8]. As far as it concerns the property (K) or (Q) we refer to papers of Cesari [4] and [5], Olech [20] and to references in there. All these connections are pointed out in remarks which are made after the definitions we give or after the statements generalizing some known results.

Our approach is based on several facts:

1° All the mentioned above modes of convergence of a net $\{K_\iota\}_J \subset \mathscr{K}(F)$ can be defined by means of the equality

$$(*)\text{-}\liminf_J K_\iota = (*)\text{-}\limsup_J K_\iota \quad (* \text{ stands for } K \text{ or } Q \text{ or } o)$$

with properly defined $(*)$-$\liminf K_\iota$ and $(*)$-$\limsup K_\iota$.

2° If the directed set J possesses the greatest element ω we may as well define the notions of semi-continuity (upper and lower), of continuity and also the property (K) or (Q) of the net $\{K_\iota\}_J$.

3° Following Goodman ([10]) we introduce the structures of complete lattices in the classes $\mathscr{K}(F)$, $\Gamma(F)$ and $\Gamma(G)$ so that the bijections: $\mathscr{K}(F) \ni K_f \to f \in \Gamma(F)$, $\Gamma(F) \ni f \to f^* \in \Gamma(G)$ (f^* denotes the conjugate of f) are anti-isomorphisms and the bijection $\mathscr{K}(F) \ni K_f \to f \in \Gamma(G)$ is isomorphism of the lattices in question. This enables us to transfer all the notions and statements made for a net $\{K_{f_\iota}\}_J$ in $\mathscr{K}(F)$ into the appropriate notions and statements formulated for a net $\{f_\iota\}_J$ in $\Gamma(F)$, as well as to get some interesting characterizations in terms of the net $\{f_\iota^*\}_J$ in $\Gamma(G)$.

After the preliminaries given in Section 2 and the definitions of Section 3, in Section 4 we formulate basic properties of the defined notions. Section 5 contains the main results. Theorem 5.1 may be considered (compare Section 5.1) as an extension (to the case of a net in the locally convex spaces) of Cesari's characterization of the property (Q) of an orientor field by the weak seminormality condition (see Cesari [5], Goodman [10]). Theorem 5.2 generalizes (compare Remark 5.7 and Corollary 5.2) the result of Boccardo and Marcellini ([1] Theorem 3.1) that under some growth condition the G-convergence of a sequence of convex and lower semi-continuous functions on a reflexive and separable Banach space is equivalent to the convergence in Kuratowski sense ((K)-convergence) of their epigraphs. In Theorem 5.2 the growth condition is relaxed and the assumption of separability is not actually needed. It is worth noticing that the extended weak seminormality (EWS for short) condition of Theorem 5.1 is not sufficient to assure the assertion of Theorem 5.2; it is proved by Example 5.1. The same example shows (see Remark 5.1 and 5.3) that the (W) (or $(\widetilde{\mathrm{W}})$) condition is indispensable for Theorem 5.2 and Theorem 5.3 to hold. The implication $(B)\Rightarrow(C)$ of Theorem 5.3 may be considered as the generalization of a result of Olech ([20] Proposition 3, see also Cesari [5] (9.ii)) giving a sufficient condition for the property (K) of an orientor field implies the property (Q). The proofs of these theorems are based on Lemma 5.1, which is proved in Section 6, and on a characterization of the (Q)-limits given in author's paper [6]. Section 6 contains also the proof of mentioned above implication $(B)\Rightarrow(C)$ and some auxiliary results. In Section 7 we pass from the net $\{K_{f_\iota}\}_J$ to the nets $\{f_\iota\}_J$ and $\{f_\iota^*\}_J$ in order to obtain some characterizations of the limits considered previously, as well as of the (Q)- and (o)-convergences (compare Proposition 7.5 and Corollary 7.1). Some results that we obtain generalize similar statements of Marcellini (see Remark 7.3). Finally, in Section 8 we compare the (o)-convergence to the notion of convergence in the sense of Mosco. Theorem 8.1 also generalizes a result of Marcellini — see Remark 8.2.

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2. Preliminaries

For most of the notions defined below, as well as for their properties we refer to Bourbaki [2], Ekeland and Temam [7], Goodman [10] and Rockafellar [22].

2.1. Let F and G be real vector spaces in duality with respect to a bilinear form as in the Introduction (see also [2] p. 48). Each space is endowed with locally convex topology that makes the other into its topological dual with respect to the pairing. Such topologies, which are evidently Hausdorff, are said to be compatible with this duality (or with the pairing). Any such topologies will do for the theory, since they all give the same closed convex sets. To fix an idea we confine ourselves to the pair of weak topologies $\sigma(F, G)$ and $\sigma(G, F)$ which are evidently compatible with the duality.

The Cartesian products $F\times \boldsymbol{R}$ and $G\times \boldsymbol{R}$ are placed in duality by the bilinear form

$$\langle (x, \mu), (y, \nu)\rangle = \langle x, y\rangle + \mu\nu$$

and the weak topologies $\sigma(F\times \boldsymbol{R}, G\times \boldsymbol{R})$ and $\sigma(G\times \boldsymbol{R}, F\times \boldsymbol{R})$ are equivalent, respectively, to the product topologies $\sigma(F, G)\times\tau_{\boldsymbol{R}}$ and $\sigma(G, F)\times\tau_{\boldsymbol{R}}$, where $\tau_{\boldsymbol{R}}$ denotes the natural topology on the reals $\boldsymbol{R}$.

2.2. Given a subset P of F, we define its polar P^0 by

(2.1) $$P^0 = \{y\in G \mid \langle x, y\rangle \leqslant 1 \text{ for every } x\in P\}.$$

The polar P^0 is convex and $\sigma(G, F)$-closed subset of G. If P is a cone; i.e. $\lambda P\subset P$ for $\lambda\geqslant 0$, then in (2.1) 1 may be replaced by 0. Let us put

(2.2) $$C_P = \{c\in F \mid p+\lambda c\in P \text{ for all } p\in P \text{ and } \lambda>0\},$$

C_P is called the asymptotic cone of the set P. It is easy to observe that if P is convex and closed in a topology τ not weaker than $\sigma(F, G)$, so is the asymptotic cone C_P. Directly from the definitions above we obtain

(2.3) $$P^0\subset (C_P)^0 =: C_P^0.$$

2.3. Following Klee and Olech [13] we call a class $\mathscr{A} = \mathscr{A}(F)$ of subsets of F admissible, provided that it satisfies the following conditions:

(A_1) Every member A of $\mathscr{A}$ is $\sigma(F, G)$-bounded; that is $\sup_{x\in A}\langle x, y\rangle < +\infty$ for all $y\in G$.

(A_2) $\mathscr{A}$ includes the convex hulls, the $\sigma(F, G)$-closures and all subsets of its members.

(A_3) The union of two members of $\mathscr{A}$ is a member of $\mathscr{A}$.

(A_4) If $A\in\mathscr{A}$, $p\in F$ and λ is a non-zero real number, then $p+\lambda A\in\mathscr{A}$.

(A_5) F is covered by $\mathscr{A}$.

With $\mathscr{A}$ as described, $\mathscr{T}_{\mathscr{A}}$ denotes the topology (for G) of uniform convergence on members of $\mathscr{A}$. Thus $(G, \mathscr{T}_{\mathscr{A}})$ is a locally convex space in which a basis of neighbourhoods of the origin 0 is formed by the class $\{A^0 \mid A\in\mathscr{A}\}$ of all the polars of members of $\mathscr{A}$.

It is easily seen that the class

$$\mathscr{A}_1 = \mathscr{A}_1(F) = \{A \subset F \,|\, A \text{ is } \sigma(F, G)\text{-relatively compact}\}$$

is admissible and the $\mathscr{T}_{\mathscr{A}_1}$ topology on G is stronger than Mackey topology $\tau(G, F)$ of uniform convergence on all convex, balanced and $\sigma(F, G)$-compact sets of F (shortly $\mathscr{T}_{\mathscr{A}_1} \succ \tau(G, F)$). Similarly, the class

$$\mathscr{A}_2 = \mathscr{A}_2(F) = \{A \subset F \,|\, A \text{ is } \sigma(F, G)\text{-bounded}\}$$

is also admissible and $\mathscr{T}_{\mathscr{A}}$-topology on G is stronger than $\mathscr{T}_{\mathscr{A}_1}$ (i.e. $\mathscr{T}_{\mathscr{A}_2} \succ \mathscr{T}_{\mathscr{A}_1}$). Bourbaki [2] calls it the strong topology on G.

By Bourbaki's Proposition 4 ([2], IV, § 3) and Proposition 5 ([2], IV, § 2) we get immediately

PROPOSITION 2.1. *If a locally convex and Hausdorff space F is semi-reflexive and $G = F'$ denotes the topological dual of F, then $\tau(G, F) = \mathscr{T}_{\mathscr{A}_2} = \mathscr{T}_{\mathscr{A}_1}$.*

Since from Theorem 1 and Theorem 2 of Bourbaki ([2], IV, § 3) it follows that every semi-reflexive normed space is a reflexive Banach space, we have

PROPOSITION 2.2. *If F is a reflexive Banach space and G denotes its topological dual F', then the assertion of Proposition* 2.1 *holds and all these topologies coincide with topology given by the norm in F.*

2.4. A function $f\colon F \to \overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$ is called regular (with respect to the pairing) iff it can be written as the supremum of a family of affine functions; that is

$$f(x) = \sup\{\langle x, y_\iota\rangle + c_\iota\} \quad \text{for all } x \in F,$$

where $y_\iota \in G$, $c_\iota \in R$ and ι runs over some index set possibly empty. If the latter is the case then we put $f(x) \equiv -\infty$. Notice that a regular function cannot take the value $-\infty$ without being identically equal to $-\infty$.

We adopt the notations:

$$\Gamma(F) = \{f\colon F \to \overline{\boldsymbol{R}} \,|\, f \text{ is regular}\}, \qquad \Gamma_0(F) = \{f \in \Gamma(F) \,|\, f \not\equiv -\infty,\, f \not\equiv +\infty\}$$
$$\mathscr{K}(F) = \{\operatorname{epi} f \,|\, f \in \Gamma(F)\}, \qquad \mathscr{K}_0(F) = \{\operatorname{epi} f \,|\, f \in \Gamma_0(F),$$

where the epigraph of a function $f\colon F \to \overline{\boldsymbol{R}}$ is the set in $F \times \boldsymbol{R}$ defined by the formula

$$\operatorname{epi} f = \{(x, \mu) \in F \times \boldsymbol{R} \,|\, \mu \geqslant f(x)\}.$$

A function $f\colon F \to \overline{\boldsymbol{R}}$ is convex iff the set $\operatorname{epi} f$ is convex (i.e. with any two points of the set all segment lies in it), and f is lower semi-continuous (l.s.c.) with respect to the topology $\sigma(F, G)$ iff the set $\operatorname{epi} f$ is closed with respect to the topology $\sigma(F \times \boldsymbol{R}, G \times \boldsymbol{R})$. Directly from the above definitions we obtain the inclusions:

$$\Gamma(F) \subset \{f\colon F \to \overline{\boldsymbol{R}} \,|\, f \text{ is convex and l.s.c.}\},$$
$$\mathscr{K}(F) \subset \{K \subset F \times \boldsymbol{R} \,|\, K \text{ is convex and } (\sigma(F, G) \times \tau_{\boldsymbol{R}})\text{-closed}\}.$$

The domain of a convex function f as above we define by the formula

$$\operatorname{dom} f = \{x \in F \mid f(x) \text{ is finite}\}.$$

There is a natural one to one correspondence (bijection) between the elements of $\Gamma(F)$ and $\mathscr{K}(F)$, namely

$$\varkappa_F:\ \Gamma(F) \ni f \to K_f := \operatorname{epi} f \in \mathscr{K}(F). \tag{2.4}$$

In particular we have

$$f \equiv -\infty \mapsto K_f = F \times \boldsymbol{R}$$
$$f = +\infty \mapsto K_f = \emptyset.$$

Remark 2.1. All these definitions can be repeated for the space G, so we obtain $\Gamma(G)$, $\mathscr{K}(G)$, $\varkappa_G$.

2.5. The Young–Fenchel transform, which maps the function f into its conjugate f^* given by the formula

$$f^*(y) = \sup_{x \in F} \{\langle x, y\rangle - f(x)\}, \qquad y \in G$$

is also bijective. We write it down as the map

$$\psi:\ \Gamma(F) \ni f \to f^* \in \Gamma(G). \tag{2.5}$$

Since $f^{**} = f$ for $f \in \Gamma(F)$, we have $\psi^{-1}(g) = g^*$ for $g \in \Gamma(G)$, where

$$g^*(x) = \sup_{y \in G} \{\langle x, y\rangle - g(y)\}.$$

2.6. The classes $\Gamma(F)$ and $\mathscr{K}(F)$ (the same holds for $\Gamma(G)$ and $\mathscr{K}(G)$) ordered, respectively, by the relation

$$f_1 \leqslant f_2 \quad \text{iff } f_1(x) \leqslant f_2(x) \quad \text{for every } x \in F$$

and by the inclusion become complete lattices (see Goodman [10]) when the lattice operations are defined by the formulae:

$$(\bigwedge_J f_\iota)(x) = \sup\{h(x) = \langle x, y\rangle + c \mid y \in G,\ c \in \boldsymbol{R},\ h \leqslant f_\iota\ \forall \iota \in J\} \tag{2.6}$$

$$(\bigvee_J f_\iota)(x) = \sup\{f_\iota(x) \mid \iota \in J\} \tag{2.7}$$

and

$$\bigwedge_J K_\iota = \bigcap_J K_\iota, \qquad \bigvee_J K_\iota = \operatorname{cl} \operatorname{co} \bigcup_J K_\iota. \tag{2.8}$$

Above J denotes a fixed set of indices, co stands for convex hull and cl for so-called closure in the class $\mathscr{K}$; i.e. the operation $\operatorname{cl} = \operatorname{cl}_{\mathscr{K}}$ is defined for a convex set $A \subset F \times \boldsymbol{R}$ by

$$\operatorname{cl}_{\mathscr{K}} A = \begin{cases} \text{closure of } A, \text{ if } A \text{ does not contain a vertical line} \\ F \times \boldsymbol{R}, \text{ otherwise} \end{cases} \tag{2.9}$$

(usually we shall write cl instead of $\operatorname{cl}_{\mathscr{K}}$ for simplicity sake).

As Goodman ([10]) observed the bijection $\varkappa_F$ and ψ given by (2.4) and (2.5) are anti-isomorphisms of lattices in question; i.e. we have

$$f_1 \leqslant f_2 \Rightarrow \varkappa_F(f_1) = K_{f_1} \supset K_{f_2} = \varkappa_F(f_2),$$
$$\varkappa_F(\bigwedge f_\iota) = K_{\bigwedge f_\iota} = \bigvee K_{f_\iota} = \bigvee \varkappa_F(f_\iota),$$
$$\varkappa_F(\bigvee f_\iota) = K_{\bigvee f_\iota} = \bigwedge K_{f_\iota} = \bigwedge \varkappa_F(f_\iota),$$

and similarly

$$f_1 \leqslant f_2 \Rightarrow \psi(f_1) = f_1^* \geqslant f_2^* = \psi(f_2),$$
$$\psi(\bigwedge f_\iota) = (\bigwedge f_\iota)^* = \bigvee f_\iota^* = \bigvee \psi(f_\iota),$$
$$\psi(\bigvee f_\iota) = (\bigvee f_\iota)^* = \bigwedge f_\iota^* = \bigwedge \psi(f_\iota),$$

where the lattice operations are taken for an arbitrary index set J. The last two equalities are the well known Brøndsted–Fenchel formulae (see [7], [10]).

Because of the above relation the bijection

$$\psi \circ \varkappa_F^{-1}\colon \mathscr{K}(F) \mapsto \Gamma(G)$$

is an isomorphism of these lattices.

2.7. The above considerations concerning the classes $\Gamma(F)$ and $\mathscr{K}(F)$ can be extended for the larger classes $\tilde{\Gamma}(F)$ and $\tilde{\mathscr{K}}(F)$ defined by

$$\tilde{\Gamma}(F) = \{f\colon F \to \overline{\mathbf{R}} \mid f \text{ is l.s.c.}\},$$
$$\tilde{\mathscr{K}}(F) = \{\operatorname{epi} f \mid f \in \tilde{\Gamma}(F)\}.$$

$\tilde{\Gamma}(F)$ and $\tilde{\mathscr{K}}(F)$ ordered in the same way as $\Gamma(F)$ and $\mathscr{K}(F)$ become the complete lattices under the lattice operations given by (2.7) and the following formula

(2.10) $$(\tilde{\bigwedge_J} f_\iota)(x) = \sup\{h(x) \mid h \text{ is l.s.c., } h \leqslant f_\iota \ \forall \iota \in J\},$$

and, respectively, by

(2.11) $$\bigwedge_J K_\iota = \bigcap_J K_\iota, \quad \tilde{\bigvee_J} K_\iota = \operatorname{cl} \bigcup_J K_\iota.$$

Above cl means the usual operation of closure (it will be called sometimes the closure in the class $\tilde{\mathscr{K}}$).

Now, the mapping $\varkappa_F$ defined by (2.4) may be considered as the restriction to $\Gamma(F)$ of the following bijection

(2.12) $$\tilde{\varkappa}_F\colon \tilde{\Gamma}(F) \ni f \mapsto K_f := \operatorname{epi} f \in \tilde{\mathscr{K}}(F).$$

The map $\tilde{\varkappa}_F$ is also an anti-isomorphism of the lattices $\tilde{\Gamma}(F)$ and $\tilde{\mathscr{K}}(F)$. However, it cannot be said that anti-isomorphism $\varkappa_F$ is the restriction of that $\tilde{\varkappa}_F$ for the lattice operations $\tilde{\bigwedge}$ and $\tilde{\bigvee}$ in $\tilde{\Gamma}(F)$ and $\tilde{\mathscr{K}}(F)$ differ from $\bigwedge$ and $\bigvee$ in $\Gamma(F) \subset \tilde{\Gamma}(F)$ and in $\mathscr{K}(F) \subset \tilde{\mathscr{K}}(F)$.

Similarly, the map ψ defined by (2.5) may be considered as the restriction to $\Gamma(F)$ of the mapping

$$\tilde{\psi}\colon \tilde{\Gamma}(F) f \to f^* \in \Gamma(G) \subset \tilde{\Gamma}(G)\,, \tag{2.13}$$

where f^* is the conjugate of f. But the map $\tilde{\psi}$ is merely a "pseudo anti-isomorphism" of the lattices $\tilde{\Gamma}(F)$ and $\tilde{\mathscr{K}}(F)$ because of the following formulae:

$$\tilde{\psi}(\tilde{\bigwedge} f_\iota) = (\tilde{\bigwedge} f_\iota)^* = \bigvee f_\iota^* = \bigvee \tilde{\psi}(f_\iota) \tag{2.14}$$

$$\tilde{\psi}(\bigvee f_\iota) = (\bigvee f_\iota)^* \leqslant \tilde{\bigwedge} f_\iota^* = \tilde{\bigwedge} \tilde{\psi}(f_\iota)\,. \tag{2.15}$$

To prove these analogues of Brøndsted–Fenchel formulae notice that for any index set J and each $\iota \in J$ we have

$$\tilde{\bigwedge} f_\iota \leqslant f_\iota, \quad f_\iota \leqslant \bigvee f_\iota\,.$$

Hence we obtain

$$(\tilde{\bigwedge} f_\iota)^* \geqslant \bigvee f_\iota^*, \quad \tilde{\bigwedge} f_\iota^* \geqslant (\bigvee f_\iota)^*\,. \tag{2.16}$$

Then putting in (2.16) f_ι^* instead of f_ι and applying the Young–Fenchel transform we get the inequalities:

$$(\tilde{\bigwedge} f_\iota^*)^{**} \leqslant (\bigvee f_\iota^{**})^*, \quad (\tilde{\bigwedge} f_\iota^{**})^* \leqslant (\bigvee f_\iota^*)^{**}\,. \tag{2.17}$$

Owing to implications

$$f_\iota^{**} \leqslant f_\iota \Rightarrow \bigvee f_\iota^{**} \leqslant \bigvee f_\iota, \quad \tilde{\bigwedge} f_\iota^{**} \leqslant \tilde{\bigwedge} f_\iota$$
$$\bigvee f_\iota^* \in \Gamma(F) \Rightarrow (\bigvee f_\iota^*)^{**} = \bigvee f_\iota^*$$

the equality (2.14) follows from the second inequality of (2.17) and the first one of (2.16). The inequality (2.15) is the consequence of the second inequality of (2.16). It is clear that the equality need not hold in this case.

2.8. For a function $f\colon F \to \overline{R}$ we define its l.s.c.-regularization $\tilde{\Gamma}[f]$ (regularization in class $\tilde{\Gamma}(F)$) by means of the formula

$$\operatorname{epi} \tilde{\Gamma}[f] = \operatorname{cl}(\operatorname{epi} f)\,,$$

and similarly, its convex and l.s.c. regularization $\Gamma[f]$ (regularization in class $\Gamma(F)$) by the formula

$$\operatorname{epi} \Gamma[f] = \operatorname{cl}_{\mathscr{K}}(\operatorname{epi} f)\,.$$

So $\tilde{\Gamma}[f] \in \tilde{\Gamma}(F)$ and $\Gamma[f] \in \Gamma(F)$ for any f as above

2.9. For the sake of completeness we recall the notion of directed set and related facts (see also Frolik [8]). A binary relation $\succ$ directs a set $J \neq \emptyset$ iff it is reflexive and transitive and for any ι, $\mu \in J$ there is a $\nu \in J$ such that $\nu \succ \iota$, $\nu \succ \mu$. The set J endowed with such a relation is called a directed set. Without any loss of generality we may assume that J possesses the greatest element ω; i.e. $\omega \succ \iota$ for every $\iota \in J$, so we have $J = I \cup \{\omega\}$, $\omega \notin I$. We shall write also $\bar{I} = I \cup \{\omega\}$ and J will be used for both $J = \bar{I}$ or $J = I$.

By a net (generalized sequence) $m = \{M_\iota\}_J$ in X (shortly $m \subset X$) we mean a function from a directed set J into X. For a net $\pi = \{\pi_\nu\}_N$ in a set Y and a directed subset J of Y we admit the definitions:

π is cofinal in J iff for every $\iota_0 \in J$ there is $\nu_0 \in N$ such that $\pi(\{\nu \in N | \nu \succ \nu_0\}) \subset \{\iota \in J | \iota \succ \iota_0\}$. Given $J' \subset J$ we say that J' is a cofinal subset of J iff the net $\pi = i = \{I_\iota\}_{J'}$ ($N = J'$, I = identity) is cofinal in J. In the case $J = \bar{I}$ we agree in addition to call cofinal in $\bar{I}$ also each subset I' of I which is cofinal in I, so a cofinal subset $\bar{I}'$ of $\bar{I}$ need not contain the element ω. The family of all cofinal subsets of J will be denoted by $\mathscr{J}$. A cofinal subset J' of J is called a residual subset of J iff $J \setminus J'$ is not a cofinal subset of J. Similarly, the net π cofinal in J is called residual in J iff for every $\nu_0 \in N$ the set $J \cap \{\pi(\nu) | \nu \succ \nu_0\}$ is a residual subset of J.

Now, given two nets $m = \{M_\iota\}_J$ in X and $\pi = \{\pi_\nu\}_N$ such that $\pi(N) \subset J$ we say that the net $m \circ \pi = \{M_{\pi_\nu}\}$ is, respectively, a cofinal subnet of m iff π is cofinal in J, or a residual subnet of m iff π is residual in J.

Finally, we say that a net $m = \{M_\iota\}_J$ in a topological space X is convergent to an element $M_0 \in X$ iff for each neighbourhood U of M_0 the set $J_U = \{\iota \in J | M_\iota \in U$ is residual in J; we write then $M_0 = \lim\limits_J M_\iota$. This amounts to saying that for every U as above there is $\iota_U \in J$ such that $M_\iota \in U$ for $\iota \succ \iota_U$.

3. The main definitions

3.1. Let X be a topological space and let 2^X denote the set of all subsets of X. Consider a net $\{M_\iota\}_J$ in 2^X. We define the upper and lower (K)-limits of $\{M_\iota\}_J$ as follows:

Definition 3.1. $(K)-\limsup\limits_J M_\iota = \{x \in X |$ for every neighbourhood U of x, J_U is a *cofinal subset* of $J\}$,

(K)-$\liminf\limits_J M_\iota = \{x \in X |$ for every U as above J_U is a residual subset of $J\}$, where $J_U = \{\iota \in J | U \cap M_\iota \neq \emptyset\}$.

We quote the following two propositions (see Frolik [8]):

Proposition 3.1. *The following two conditions are equivalent*:

(i) $x \in (K)$-$\limsup\limits_J M_\iota$ $\quad$ ($x \in (K)$-$\liminf\limits_J M_\iota$)

(ii) *there exist a cofinal (residual) subnet* $\{M_{\pi_\nu}\}_N$ *of* $\{M_\iota\}_J$ *and* $x_\nu \in M_{\pi_\nu}$ *such that* $x = \lim\limits_N x_\nu$.

Proposition 3.2. *The following formulae hold*:

$$(K)\text{-}\limsup_J M_\iota = \bigcap_{\lambda \neq \omega} \operatorname{cl} \bigcup_{\iota \succ \lambda} M_\iota, \tag{3.1}$$

$$(K)\text{-}\liminf_J M_\iota = \bigcap_{J' \in \mathscr{J}} (K)\text{-}\limsup_{J'} M_\iota, \tag{3.2}$$

$$\operatorname{cl} \bigcup_{\lambda \neq \omega} \bigcap_{\iota \succ \lambda} M_\iota \subset (K)\text{-}\liminf_J M_\iota, \tag{3.3}$$