ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DLXXX PRACE MATEMATYCZNE, ZESZYT 22 1981

Absolutely continuous invariant measures for transformations on the real line

by Marian JABŁOŃSKI and Andrzej LASOTA

1. Introduction. The purpose of the present paper is to prove the existence of absolutely continuous, ergodic invariant measures for a family mappings of the real line into itself. This family contains as a special case the functions of the form $T(x) = a\operatorname{tg}(bx+c)$ for which the existence of an absolutely continuous invariant measure and ergodicity was proved by J. H. B. Kemperman [4], [7] and F. Schweiger [5]. Analogical results with more restrictive conditions were also obtained by J. H. B. Kemperman [6].

In the proof of the existence we shall use the technique developed in [2]. It is based on the fact that the Frobenius–Perron operator corresponding to the point transformation under consideration has the property of shrinking the variation of the functions. In the proof of the uniqueness we shall follow the ideas of T. Y. Li and J. Yorke [3] concerning the density functions of bounded variation.

2. Existence theorem. Let $\{I_k\}_{k=-\infty}^{k=\infty}$ be a countable partition of the real line R such that

(i) each I_k is an open set,

(ii) $I_k \cap I_j = \emptyset$ for $k \neq j$,

(iii) $R \setminus \bigcup_{k=-\infty}^{k=\infty} I_k$ is a countable set,

(iv) $\sup |I_k| = L < \infty$,

where $|I_k|$ denotes the length of I_k.

Let $T\colon \bigcup I_k \to R$ be a function satisfying the following conditions

(v) for any k the restriction T_k of T to the interval I_k is differentiable and its derivative T_k' is locally Lipschitzean

(vi) $|T_k'(x)| \geq q > 1$ for $x \in I_k$,

(vii) $T_k(I_k) = R$,

(viii) $\dfrac{|T_k(x)|}{(T_k'(x))^2} \leq M < \infty$,

(ix) $\omega(x) = \sup\limits_k \dfrac{|\Psi_k'(x)|}{I_k}$ is integrable on R, where $\Psi_k = T_k^{-1}$.

The foregoing conditions are satisfied, for example, for $T(x) = a\,\mathrm{tg}(bx+c)$ if $ab>1$. In fact, we have

$$|T'(x)| = \frac{|ab|}{\cos^2(bx+c)} \geqslant |ab| > 1\,,$$

$$\frac{|T''(x)|}{|T'(x)|^2} = \left|\frac{1}{a}\sin 2(bx+c)\right| \leqslant \frac{2}{|a|}\,,$$

$$\omega(x) = \frac{|ab|}{\pi\big(1+(bx+c)^2\big)}\,.$$

THEOREM 1. *Assume that* $T\colon \bigcup I_k \to R$ *is a function satisfying* (i)–(ix). *Then there exists on* R *a finite absolutely continuous measure* μ *invariant with respect to* T. *The density* $g = d\mu/dx$ *may be found by the formula*

$$g = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k f, \tag{1}$$

where f *is an arbitrary integrable function and*

$$Pf(x) = \frac{d}{dx}\int\limits_{T^{-1}(-\infty,x)} f(s)\,ds\,. \tag{2}$$

Proof. From the definition of P it follows immediately that any integrable function g satisfying $Pg = g$ represents the density of a finite invariant measure. Moreover the limit function g given by formula (1) is automatically a fixed point of P. Thus we need only to prove that the limit (1) exists.

For $f\in L^1(R)$ and $f\geqslant 0$ we have

$$\int\limits_{T^{-1}(-\infty,x)} f(s)\,ds = \int\limits_{\bigcup_k T_k^{-1}(-\infty,x)} f(s)\,ds = \sum_k \int\limits_{T_k^{-1}(-\infty,x)} f(s)\,ds$$

$$= \sum_k \int\limits_{-\infty}^{x} f\big(\Psi_k(s)\big)|\Psi_k'(s)|\,ds = \int\limits_{-\infty}^{x} \sum_k f\big(\Psi_k(s)\big)|\Psi_k'(s)|\,ds$$

and, consequently

$$Pf(x) = \sum_k f\big(\Psi_k(x)\big)|\Psi_k'(x)|\,.$$

Since the last series is absolutely convergent for almost all x and $Pf = Pf^+ - Pf^-$ where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ we obtain

$$Pf(x) = \sum_k f\big(\Psi_k(x)\big)|\Psi_k'(x)| \tag{3}$$

for any $f\in L^1(R)$.

For given f we can choose points $z_k\in I_k$ such that

$$\sum_k |I_k|\,|f(z_k)| \leqslant \int\limits_{-\infty}^{+\infty} |f(x)|\,dx\,.$$

Consequently

$$\sum_k |I_k|\,|f(\Psi_k(x))| \leqslant \sum_k |I_k|\,|f(\Psi_k(x))-f(z_k)| + \sum_k |I_k|\,|f(z_k)| \leqslant L \bigvee_{-\infty}^{+\infty} f + \int_{-\infty}^{+\infty} |f(x)|\,dx\,,$$

where $\bigvee_{-\infty}^{+\infty} f$ denotes the variation of f on R. Using the last inequality we obtain

$$|Pf| \leqslant P|f| \leqslant \omega\Big(L \bigvee_{-\infty}^{+\infty} f + \int_{-\infty}^{+\infty} |f(x)|\,dx\Big). \tag{4}$$

Now we are going to evaluate the variation of Pf. We have

$$\bigvee_{-\infty}^{+\infty} Pf \leqslant \sum_k \bigvee_{-\infty}^{+\infty} (f\circ\Psi_k)|\Psi_k'| = \sum_k \int_{-\infty}^{+\infty} |d((f\circ\Psi_k)|\Psi_k'|)|$$

$$\leqslant \sum_k \int_{-\infty}^{+\infty} |\Psi_k'|\,|d(f\circ\Psi_k)| + \sum_k \int_{-\infty}^{+\infty} |f\circ\Psi_k|\,|d\Psi_k'|\,.$$

From (vi) and (viii) it follows that

$$|\Psi_k'| = \frac{1}{|T_k'|} \leqslant \frac{1}{q} \quad \text{and} \quad \frac{|\Psi_k''|}{|\Psi_k'|} = \frac{|T_k''|}{|T_k'|^2} \leqslant M\,.$$

Therefore

$$\bigvee_{-\infty}^{+\infty} Pf \leqslant \frac{1}{q} \sum_k \int_{-\infty}^{+\infty} |d(f\circ\Psi_k)| + M \sum_k \int_{-\infty}^{+\infty} |f\circ\Psi_k|\,|d\Psi_k|$$

and consequently

$$\bigvee_{-\infty}^{+\infty} Pf \leqslant \frac{1}{q} \sum_k \int_{I_k} |df| + M \sum_k \int_{I_k} |f|\,dx = \frac{1}{q} \bigvee_{-\infty}^{+\infty} f + M \int_{-\infty}^{+\infty} |f|\,dx\,.$$

Using this and the obvious inequality

$$\int_{-\infty}^{+\infty} |Pf|\,dx \leqslant \int_{-\infty}^{+\infty} P|f|\,dx = \int_{-\infty}^{+\infty} |f|\,dx\,,$$

we obtain

$$\bigvee_{-\infty}^{+\infty} P^n f \leqslant \frac{1}{q^n} \bigvee_{-\infty}^{+\infty} f + \frac{qM}{q-1} \int_{-\infty}^{+\infty} |f|\,dx \qquad (n = 1, 2, \ldots)\,. \tag{5}$$

Finally, from (4) and (5) it follows that

$$|P^n f| \leqslant \omega\Big(\frac{L}{q^{n-1}}\Big) \bigvee_{-\infty}^{+\infty} f + \Big(1 + \frac{qM}{q-1}\Big) \int_{-\infty}^{+\infty} |f|\,dx\,.$$

Therefore for any integrable function f of bounded variation there exists a constant K_f such that

$$|P^n f(x) \leqslant K_f \omega(x) .$$

Since ω is integrable, this implies that the sequence $\{P^n f\}$ is weakly compact in $L^1(R)$. The set of functions of bounded variation is dense in $L^1(R)$ and consequently, according to the Yosida–Kakutani ergodic theorem, the sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} P^k f\right\}$$

converges strongly for each $f \in L^1(R)$. This completes the proof.

Remark 1. It can be easily shown that under condition of Theorem 1 the variation of the limit function g in (1) is finite for any integrable f. In fact, from (5) and (1) it follows that

$$\bigvee_{-\infty}^{+\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} P^k f\right) \leqslant \frac{1}{n}\frac{q}{q-1}\bigvee_{-\infty}^{+\infty} f + \frac{qM}{q-1}\int_{-\infty}^{+\infty} |f|\,dx .$$

From this, using Helly's theorem, we obtain

$$\bigvee_{-\infty}^{+\infty}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} P^k f\right) \leqslant \frac{qM}{q-1}\int_{-\infty}^{+\infty} |f|\,dx \tag{6}$$

for each function f of bounded variation. Now we may apply Helly's theorem once more to extend inequality (6) to the closure of the set of functions of bounded variation, that is to all of $L^1(R)$.

3. Uniqueness theorem. Now we are going to show that the measure $d\mu = g\,dx$ given by formula (1) is unique. We start from some simple lemmas which are not related with the specific form of the function T in Theorem 1. Lemmas 1 and 3 are due to T. Y. Li and J. Yorke [3].

A transformation $T\colon R \to R$ will be called nonsingular if $T(E)$ and $T^{-1}(E)$ are measurable for each Borel set $E \subset R$ and if

$$m(E) = 0 \Rightarrow m(T(E)) = m(T^{-1}(E)) = 0 ,$$

where m denotes the standard Borel measure on R. Given an absolutely continuous measure μ on R we shall denote by $\Delta\mu$ the support of μ, that is

$$\Delta_\mu = \{x\colon g(x) > 0\} , \quad \text{where } g = \frac{d\mu}{dx} .$$

We shall write

$$A = B(\mathrm{mod}\,0)$$

if $m(A \div B) = 0$, where $\div$ stand for the symmetric difference of the sets A and B.

LEMMA 1. *Let T: $R \to R$ be a nonsingular transformation and let μ be a finite, absolutely continuous measure invariant with respect to T. Then*

(7) $$\Delta_\mu = T(\Delta_\mu)(\mathrm{mod}\,0).$$

Proof. Since $d\mu = g\,dx$ is invariant, we have

$$\int_E g\,dx = \int_{T^{-1}(E)} g\,dx$$

for each Borel subset $E \subset R$. In the particular setting $E = \Delta_\mu$ or $E = T(\Delta_\mu)$ we obtain

$$\int_{\Delta_\mu} g\,dx = \int_{T^{-1}(\Delta_\mu)} g\,dx, \qquad \int_{T(\Delta_\mu)} g\,dx = \int_{T^{-1}(T(\Delta_\mu))} g\,dx \geqslant \int_{\Delta_\mu} g\,dx.$$

Since Δ_μ is the support of μ this implies

$$m(\Delta_\mu \setminus T^{-1}(\Delta_\mu)) = 0 \quad \text{and} \quad m(\Delta_\mu \setminus T(\Delta_\mu)) = 0.$$

From this and nonsingularity of T it follows (7).

LEMMA 2. *Let T: $R \to R$ be a nonsingular transformation such that*

$$\Delta_\mu = \Delta_\nu(\mathrm{mod}\,0)$$

for any two probabilistic absolutely continuous invariant measures μ and ν. Then T admits at most one probabilistic absolutely continuous invariant measure.

Proof. Suppose that μ_1 and μ_2 be two different invariant probabilistic measures. Denote by σ^+ and σ^- the positive and the negative part of the countably additive function $\sigma = \mu_1 - \mu_2$, that is

$$d\sigma^+ = (f_1 - f_2)^+ dx, \quad d\sigma^- = (f_1 - f_2)^- dx,$$

where f_1 and f_2 are the densities of the measures μ_1 and μ_2 respectively. Since $f_1 = f_2$ on a set of a positive measure and f_i are normalized we have

$$\int_{-\infty}^{+\infty} (f_1 - f_2)^+ dx = \int_{-\infty}^{+\infty} (f_1 - f_2)^- dx = r > 0.$$

Setting $\mu = r^{-1}\sigma^+$ and $\nu = r^{-1}\sigma^-$ we obtain two invariant probabilistic measures with disjoint supports. This contradicts our assumptions and finishes the proof.

LEMMA 3. *Let f: $R \to R$ be a function of bounded variation. Assume moreover that*

$$\int_{-\infty}^{+\infty} f(x)\,dx > 0, \quad f(x) \geqslant 0.$$

Then there exists an interval $\Delta \subset R$ such that $f(x) > 0$ for $x \in \Delta$.

Proof. Choose an $\varepsilon>0$ and an interval $[a, b]$ such that

$$\int_a^b f(x)\,dx>\varepsilon(b-a). \tag{8}$$

Moreover, suppose that the interior of the set

$$A = \{x\in[a, b]: f(x)>0\}$$

is empty, i.e.

$$\operatorname{int} A = \emptyset. \tag{9}$$

From (8) and (9) it follows that the set $B = \{x\in[a, b]: f(x)>\varepsilon\}$ contains infinitely many points and that for any x_1, x_2 belonging to B there is an $x_3\in[x_1, x_2]$ such that $f(x_3) = 0$. But this is impossible because the variation of f is bounded. Therefore $[a, b]$ contains an interval Δ on which f is different from zero.

THEOREM 2. *Assume that T: $R\to R$ is a function satisfying conditions* (i)–(ix). *Then the probabilistic, absolutely continuous measure invariant under T is unique and consequently ergodic.*

Proof. Let μ be an arbitrary invariant measure with the required properties. The existence of a measure follows from Theorem 1. Since μ is invariant its density $g = d\mu/dx$ satisfies the equation $g = Pg$, where P is the Frobenius–Perron operator defined by formula 2. Consequently

$$g = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} P^k g.$$

According to Remark 1 this implies that g is a function of bounded variation and by Lemma 3, $g(x)>0$ on an interval Δ. From conditions (i)–(vii), it follows that the domain of the function T^n $(n = 1, 2, \ldots)$ may be divided into a countable number of intervals I_k^n such that $T^n(I_k^n) = R$ and the length of I_k^n is smaller than of equal to L/q^n. Therefore $T^n(\Delta) = R$ if n is sufficiently large, namely if $2L/q^n<m(\Delta)$. Since the interval Δ is contained in the support Δ_μ of the measure μ, we have also $T^n(\Delta_\mu) = R$ for large n. According to Lemma 1, this implies $\Delta_\mu = R(\mathrm{mod}\,0)$. Thus we have proved that any probabilistic, absolutely continuous measure, invariant under T admits the same support equal to R. By Lemma 2 this implies that the measure is unique.

References

[1] L. A. Bunimowič, *On a mapping of a circle* (Russian), Mat. Zametki, 8 (1970), 205–216.

[2] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), 481–488.

[3] T. Y. Li and J. Yorke, *Ergodic transformations from an interval into itself* (preprint).

[4] J. H. B. Kemperman, *The ergodic behavior of a class of real transformations*, Stochastic Processes and Related Topics Proc. Summer Res. Inst. on Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Indiana, (1974), Academic Press, New York, (1975), 249–258.

[5] F. Schweiger, *tanx is ergodic*, Proc. Amer. Math. Soc. 1 (1978), 54–56.

[6] J. H. B. Kemperman, *The ergodic behavior of a class of real transformations* (preprint).

[7] — *Applicable sufficient conditions for the existence of a finite invariant measure for a piecewise monotone transformation*, to appear.

UNIWERSYTET JAGIELLOŃSKI
INSTYTUT INFORMATYKI
KRAKÓW (POLAND)