

## On the $(Q)$ -convergence of nets of regular convex functions

by Zdzisław DENKOWSKI

**1. Introduction.** The notion of the  $(Q)$ -convergence of a net of regular convex functions defined on a locally convex space was introduced in Goodman's paper [5] (see Remark 3.2 in [3]) and author's paper [3]. It generalizes the notion of the  $G$ -convergence defined by Marcellini [6] for a sequence of regular convex functions defined on a separable and reflexive Banach space. For the relations with other types of convergence (in the sense of Kuratowski or Mosco) we refer to [3].

It seems natural to search for the properties of the  $(Q)$ -convergence which would generalize the properties of the  $G$ -convergence obtained by Marcellini in [6] [7] and by Boccardo and Marcellini [2]. After preliminaries contained in Section 2 we give such generalizations in Section 4 (Propositions 4.2–4.7). We would like to mention that our statements concern the larger class of functions (the class  $C(\alpha, x_0, M) \subset \Gamma_0(F)$  with  $\alpha$  satisfying the relaxed growth condition — see Remark 4.3) than that considered by Marcellini (see Remark 4.2); besides, some of them are proved by quite different methods. For instance, using the characterization of the  $(Q)$ -limits given in Section 3, we prove (Theorem 3.1) that from any net of convex and closed sets one can choose a  $(Q)$ -convergent subnet. Hence, as a simple corollary we obtain Proposition 4.3 which generalizes the statement of Marcellini (Teorema 2.4 in [2]) that from any sequence  $\{f_n\}_N \subset C(\alpha, v_0, M) \subset \Gamma_0(V)$ , where  $V$  is a reflexive and separable Banach space and  $\alpha$  satisfies the growth condition (see Remark 4.2), one can choose a  $G$ -convergent subsequence. It is worth noticing that neither reflexivity nor separability of the space is needed for Theorem 3.1 and in consequence for Proposition 4.3 to hold.

The above mentioned characterization of the  $(Q)$ -limits (Prop. 3.1) is also useful to obtain the characterization (Prop. 4.1) of the  $(Q)$ -convergence by a condition similar to the so-called extended weak seminormality condition (see Def. 5.1 in [3]). In Section 5 we give an example which illustrates the theory developed in Section 4. Especially, Proposition 5.1 shows that if we relax the growth condition put on the functional (given by formula (5.9)) we minimize, we can obtain the solution of the corresponding Euler equation (5.7) by the direct method only for certain vectors from the dual space.

The author wishes to thank Professors C. Olech and A. Plis for discussing the problem.

**2. Preliminaries.** All the notions and definitions used throughout this paper are explained in details in [3] by the present author.

We recall some of them briefly. Let the locally convex spaces  $F$  and  $G$  be paired by a bilinear form  $\langle x, y \rangle$ . By  $\Gamma(F)$  we denote the family of all real convex and regular functions defined on  $F$  and  $\Gamma_0(F)$  denotes the subset of  $f \in \Gamma(F)$  other than the constants  $+\infty$  and  $-\infty$ . We adopt the notation

$$\mathcal{K}(F) = \{K_f = \text{epi } f \mid f \in \Gamma(F)\}, \quad \Gamma_0(F) = \{K_f \mid f \in \Gamma_0(F)\},$$

where the epigraph of a function  $f: F \rightarrow \mathbf{R}$  is defined as follows:

$$\text{epi } f = \{(x, \mu) \in F \times \mathbf{R} \mid \mu \geq f(x)\}.$$

By a net in  $\mathcal{K}(F)$  (or in  $\Gamma(F)$ ) we mean a function  $J \ni \iota \mapsto K_\iota \in \mathcal{K}(F)$  (or  $J \ni \iota \mapsto f_\iota \in \Gamma(F)$ ), where  $J$  denotes a directed set which may or may not possess the greatest element  $\omega$ ; i.e.  $J = I \cup \{\omega\}$  or  $J = I$ ,  $\omega \notin I$ . Such nets will be noted as  $\{K_\iota\}_J$  and  $\{f_\iota\}_J$ , respectively. Given a net  $\{K_\iota\}_J$  we shall denote by  $\{K_{\Pi_\nu}\}_N$  its cofinal or residual subnet; this means that the net  $\{\Pi_\nu\}_N$  is cofinal or residual, respectively, in the directed set  $J$  (see Section 2.9 in [3]).

The upper and lower  $(Q)$ -limits of the net  $\{K_\iota\}_J$  are defined by the formulae (see Def. 3.2 in [3]):

$$\begin{aligned} (Q)\text{-}\limsup K_\iota &= \bigcap_{\lambda \in J} \text{clco} \bigcup_{\iota \succ \lambda} K_\iota, \\ (Q)\text{-}\liminf K_\iota &= \bigcap_{J' \in \mathcal{J}} (Q)\text{-}\limsup_{J'} K_\iota, \end{aligned}$$

where  $\mathcal{J}$  is the set of all cofinal subsets of  $J$ ,  $\text{co}$  stands for convex hull and  $\text{cl} = \text{cl}_{\mathcal{K}}$  denotes the closure in the class  $\mathcal{K}$ ; i.e. for a set  $K \subset F \times \mathbf{R}$ ,  $\text{cl}_{\mathcal{K}} K$  means the  $\sigma(F, G) \times \tau_{\mathbf{R}}$ -closure in the product topology of the weak topology in  $F$  and natural topology in  $\mathbf{R}$  provided the set  $K$  does not contain a vertical line, and  $F \times \mathbf{R}$  otherwise. In the case  $J = I$  we say the net  $\{K_\iota\}_I$  is  $(Q)$ -convergent iff both  $(Q)$ -limits coincide; its common value  $K$  we call the  $(Q)$ -limes; we then write  $K = (Q)\text{-}\lim_I K_\iota$ .

**3. A characterization of the  $(Q)$ -limits and its application.** By an affine hyperplane (shortly hyperplane) in the space  $F \times \mathbf{R}$  we mean the set  $H = \{(x, \mu) \in F \times \mathbf{R} \mid \langle x, y \rangle + \mu v = \alpha\}$ , where  $(y, v)$  is a vector of  $G \times \mathbf{R}$  and  $\alpha$  is a real number. If  $F$  and  $G$  are supplied with topologies which are compatible with the duality (for instance, with the weak topologies  $\sigma(F, G)$  and  $\sigma(G, F)$ , respectively), then  $H$  is a closed subset of  $F \times \mathbf{R}$  in the product topology  $\sigma(F, G) \times \tau_{\mathbf{R}}$ ; its complement is the union of two open half-spaces. Thus, we define an open (closed) half-space in a similar way as  $H$  replacing the equality by any strong (weak) inequality; i.e. by  $>$  or  $<$  ( $\geq$  or  $\leq$ ).

Under the above notation we have:

**PROPOSITION 3.1.** *Assume  $\{K_\iota\}_J$  is a net in  $\mathcal{K}(F)$ . If  $(Q)\text{-}\limsup_J K_\iota$  does not contain a vertical line (i.e. the set  $\{x\} \times \mathbf{R}$  with some  $x \in F$ ), then the following two conditions are equivalent:*

(i)  $a \in (Q)\text{-limsup}_J K_i$  ( $a \in (Q)\text{-liminf}_J K_i$ ),

(ii) For every hyperplane  $H$  in  $F \times R$ , if we denote by  $H^a$  the open half-space containing  $a$ , then  $J_a = \{\iota \in J \mid H^a \cap K_i \neq \emptyset\}$  is a cofinal (residual) subset of  $J$ .

Proof. For the proof suppose  $J$  does not possess the greatest element (the remaining case is analogous).

(i)  $\Rightarrow$  (ii). Suppose first  $a \in (Q)\text{-limsup}_J K_i$ . Thus, for any  $\lambda \in J$  and for any neighbourhood  $\mathcal{V}_a$  of  $a$  we have

$$\mathcal{V}_a \cap \text{co} \bigcup_{i \succ \lambda} K_i \neq \emptyset.$$

Hence, setting  $\mathcal{V}_a = H^a$  we obtain that for any  $\lambda \in J$  there is  $i_\lambda \succ \lambda$  such that  $H^a \cap K_{i_\lambda} \neq \emptyset$  which means  $J_a$  is cofinal in  $J$ .

Now, let  $a \in (Q)\text{-liminf}_J K_i$ . Hence, by the first part of the proof for any  $J'$  cofinal in  $J$  there is  $J'_a = \{\iota \in J' \mid K_i \cap H^a \neq \emptyset\}$  which is cofinal in  $J'$ . We claim that  $J_1 = \bigcup_{J' \in \mathcal{J}} J'_a$  is residual in  $J$ . Indeed, if it were not, then  $J'' = J \setminus J_1$  would be cofinal in  $J$ . So there would exist  $J''_a$  cofinal in  $J'' \in \mathcal{J}$  and therefore  $J''_a$  would have to belong to  $J_1$  which is impossible for  $J''_a \subset J'' = J \setminus J_1$ . Thus,  $K_i \cap H^a \neq \emptyset$  for  $i \in J_1$  which is residual in  $J$ .

(ii)  $\Rightarrow$  (i). Suppose the contrary. We suppose first  $a \notin (Q)\text{-limsup}_J K_i$ . Hence, there is a  $\lambda \in J$  such that  $a \notin \text{clco} \bigcup_{i \succ \lambda} K_i$ . Then the point  $a$  can be strongly separated from the convex and closed set in question by a hyperplane  $H$ . Thus, if  $a$  belongs to the open half-space  $H^a$  and  $\text{clco} \bigcup_{i \succ \lambda} K_i$  is contained in the opposite open half-space  $H_1^a$ , we have

$$K_i \subset H_1^a \subset \setminus H^a, \quad i \succ \lambda.$$

In consequence

$$J \setminus J_a = \{\iota \in J \mid K_i \cap H^a = \emptyset\} \supset \{\iota \in J \mid i \succ \lambda\}.$$

Therefore,  $J_a$  is not cofinal, which contradicts (ii).

Now, suppose  $a \notin (Q)\text{-liminf}_J K$ . Hence, owing to

$$(Q)\text{-liminf}_J K \supset (K)\text{-liminf}_J K,$$

(see formula (4.5) in [3]) we find by Def. 3.1 in [3] a  $\sigma(F \times R, G \times R)$ -neighbourhood  $\mathcal{V}_a$  of the point  $a$  such that  $J_2 = \{\iota \in J \mid K_i \cap \mathcal{V}_a = \emptyset\}$  is cofinal in  $J$ . We can represent this

neighbourhood in the form  $\mathcal{V}_a = \bigcap_{r=1}^p H_r^a$  with some open halfspaces  $H_r^a (r = 1, \dots, p)$

containing the point  $a$ . Hence, by the definition of  $J_2$  we get

$$K_i \subset \bigcup_{r=1}^p (\setminus H_r^a), \quad i \in J_2,$$

and in consequence there is  $r_0 \in \{1, \dots, p\}$  such that

$$K_i \cap H_{r_0}^a = \emptyset, \quad i \in J'_2,$$

where  $J'_2$  is a cofinal subset of  $J_2$  and so it is cofinal in  $J$ . Therefore,

$$J'_a = \{\iota \in J \mid K_\iota \cap H_{r_0}^a \neq \emptyset\}$$

cannot be residual, a contradiction with the assumption (ii). Thus, the proof is completed.

**Remark 3.1.** The Proposition 3.1 provides a characterization of the  $(Q)$ -limits which is similar to that given in the definition of the limits in Kuratowski sense ( $(K)$ -limits), compare Frolik [4] and Def. 3.1 in [3].

This characterization was used in the proof of Theorem 5.1 in [3]. It is also useful in proving the following

**THEOREM 3.1.** *For any net  $\{K_i\}_J$  in  $\mathcal{K}(F)$  such that  $(Q)\text{-}\limsup_J K_i$  does not contain a vertical line there is a cofinal subnet  $\{K_{\Pi_v}\}_{\mathcal{N}}$  which is  $(Q)$ -convergent.*

**Proof.** We shall give the proof in two steps.

1° The construction of the cofinal subnet  $\{K_{\Pi_v}\}_{\mathcal{N}}$  of the net  $\{K_i\}_J$ . Let  $\mathcal{B}$  denotes the family of all convex and open (in  $\sigma(F, G) \times \tau_{\mathbf{R}}$  topology) subsets  $U$  of the space  $F \times \mathbf{R}$  such that the set  $J_U = \{\iota \in J \mid K_\iota \subset U\}$  is cofinal in  $J$ . Since  $F \times \mathbf{R}$  belong to  $\mathcal{B}$  the family  $\mathcal{B}$  is not empty. Now, let  $\mathcal{B}'$  be the maximal multiplicative subfamily of  $\mathcal{B}$ . It is easily observed that the set  $\mathcal{N} = \{(\iota, U) \in J \times \mathcal{B}' \mid K_\iota \subset U\}$  is directed by the relation

$$(\iota_1, U_1) \succ (\iota_2, U_2) \quad \text{iff} \quad \iota_1 \succ \iota_2, U_1 \subset U_2.$$

Setting  $\Pi_v = \iota$  for  $v = (\iota, U) \in \mathcal{N}$  we find that the net  $\{\Pi_v\}_{\mathcal{N}}$  is cofinal in  $J$  (compare Section 2.9 in [3]). Hence, the net  $\{K_{\Pi_v}\}_{\mathcal{N}}$  is a cofinal subnet of the net  $\{K_i\}_J$ .

2° The proof of the equality:  $(Q)\text{-}\lim_N K_{\Pi_v} = \bigcap_{U \in \mathcal{B}'} \text{cl} U$ . Here  $\text{cl} = \text{cl}_{\mathcal{X}}$  denotes the closure of the set  $U$  in the class  $\mathcal{X}$  (compare Section 2). Denoting the right-hand side of the equality by  $K$  we observe that  $K \in \mathcal{K}(F)$ . Notice, that for the proof of the required equality it is enough to show that

$$(3.1) \quad (Q)\text{-}\limsup_{\mathcal{N}} K_{\Pi_v} \subset K$$

and

$$(3.2) \quad (Q)\text{-}\liminf_{\mathcal{N}} K_{\Pi_v} \supset K.$$

To prove (3.1) let us fix  $U_1 \in \mathcal{B}'$ . Since the set  $J_{U_1} = \{\iota \in J \mid K_\iota \subset U_1\}$  is cofinal in  $J$ , we find that for any  $\iota_0 \in J$  there is  $\iota_1 \succ \iota_0$  such that  $K_{\iota_1} \subset U_1$ . Then for any  $v = (\iota, U) \succ (\iota_1, U_1) = v_1$  we have  $K_\iota \subset U_1$ . Hence, by the definition of the upper  $(Q)$ -limit we get

$$(Q)\text{-}\limsup_{\mathcal{N}} K_{\Pi_v} \subset \text{cl} U_1$$

and in consequence we obtain (3.1) for  $U_1$  was arbitrary in  $\mathcal{B}'$ . Now, observe that (3.2) is trivial if  $(Q)\text{-}\liminf_{\mathcal{N}} K_{\Pi_v} = F \times \mathbf{R}$ . So to get (3.2) it remains to prove the implication

$$(3.3) \quad a \notin (Q)\text{-}\liminf_{\mathcal{N}} K_{\Pi_v} \Rightarrow a \notin K.$$

Thus, given a point  $a \notin (Q)\text{-}\liminf K_{\Pi_v}$ , owing to Prop. 3.1 we may find an affine hyperplane  $\tilde{H}$  in  $F \times R$  such that the set  $\mathcal{N}_a = \{v \in \mathcal{N} \mid K_{\Pi_v} \cap \tilde{H}^a = \emptyset\}$  is cofinal in  $\mathcal{N}$  ( $\tilde{H}^a$  being the open half-space containing  $a$ ). It is easily seen that such a hyperplane  $\tilde{H}$  separates strongly the point  $a$  from the convex and closed set  $(Q)\text{-}\liminf K_{\Pi_v}$  and without any loss of generality we may suppose it is oblique. Translating  $\tilde{H}^a$  a bit we obtain the open half-space  $H^a \subset \tilde{H}^a$  such that

$$(3.4) \quad a \in H^a, \quad \text{cl} H^a \subset \tilde{H}^a.$$

Since the set  $\mathcal{N}'_a = \{v \in \mathcal{N} \mid (\text{cl} H^a) \cap K_{\Pi_v} = \emptyset\}$  is by force cofinal in  $\mathcal{N}$ , the set  $U_a = (\setminus \text{cl} H^a)$  (the complement of  $\text{cl} H^a$  in  $F \times R$ ) belongs to  $\mathcal{B}$ . We shall prove that  $U_a$  belongs also to  $\mathcal{B}'$ . To this end it suffices to show that  $U_a \cap U' \in \mathcal{B}'$  for every  $U' \in \mathcal{B}'$ ; i.e. that the set  $J' = \{i \in J \mid K_i \subset U_a \cap U'\}$  is cofinal in  $J$ . So let  $i_0$  be a fixed element of  $J$ . Since  $U' \in \mathcal{B}' \subset \mathcal{B}$  we can find  $i' \in J$  such that  $i' \succ i_0$ ,  $K_{i'} \subset U'$ .

Now, owing to the fact that  $\mathcal{N}'_a$  is cofinal in  $\mathcal{N}$ , we find  $(i_1, U_1) \in \mathcal{N}'_a$  such that  $(i_1, U_1) \succ (i', U')$ .

Hence,  $K_{i_1} \subset U_a$ ,  $K_{i_1} \subset U'$ , and of course we have  $i_1 \succ i' \succ i_0$ , which proves the cofinality of  $J'$  in  $J$ . In consequence  $U_a \in \mathcal{B}'$ . Hence, from the definition of  $U_a$  and from (3.4) we get

$$a \notin \text{cl} U_a \supset \bigcap_{U \in \mathcal{B}'} \text{cl} U = K,$$

which proves (3.3). This in turn proves (3.2) which together with (3.1) completes the proof of the Theorem 3.1.

**Remark 3.2.** The proof above is similar to that of Frolik's theorem (2.14 Theorem in [4]) which we quote in our notation:

**THEOREM 3.2.** *If the topological space  $(X, \tau)$  is regular, then for any net  $\{M_i\}_J$  in  $2^X$  there is a cofinal subnet  $\{M_{\Pi_v}\}_{\mathcal{N}}$  which is  $(K)_\tau$ -convergent (compare Definition 3.6 in [3]).*

**COROLLARY 3.1.** *Since  $F \times R$  is regular with respect to the  $\sigma(F, G) \times \tau_R$  topology, the Theorem 3.2 holds for a net  $\{K_i\}_J$  in  $\mathcal{K}(F)$ .*

**4. The properties of the  $(Q)$ -convergence of a net of functions.** All the notions and properties formulated in the case of the net  $\{K_i\}_J$  in  $\mathcal{K}(F)$  may be transferred into appropriate notions and properties formulated for the net  $\{f_i\}_J$  in  $\Gamma(F)$  and *vice versa* by means of the anti-isomorphism

$$\kappa_F: \Gamma(F) \ni f \mapsto K_f = \text{epi } f \in \mathcal{K}(F)$$

of the lattices in question (see Section 2.6 and 7 in [3]). Thus, for  $J = I$  we have

$$f = (Q)\text{-}\lim_I f_i \Leftrightarrow K_f = (Q)\text{-}\lim_I K_{f_i},$$

where the left hand side is defined (see Section 7.2 in [3]) by the formulae

$$(4.1) \quad f = (Q)\text{-}\liminf_I f_i = (Q)\text{-}\limsup_I f_i,$$

$$(4.2) \quad (Q)\text{-}\liminf_I f_i = \bigvee_{\lambda \in I} \bigwedge_{i > \lambda} f_i, \quad (Q)\text{-}\limsup_I f_i = \bigvee_{I' \in \mathcal{J}} (Q)\text{-}\liminf_{I'} f_i,$$

and the lattice operations (for any set of indices) are given by

$$(4.3) \quad (\bigvee f_i)(x) = \sup f_i(x), \quad (\bigwedge f_i)(x) = \sup \{h(x) = \langle x, y \rangle + \alpha \mid y \in G, \alpha \in \mathbf{R}, h \leq f_i\}.$$

Directly from the definition we find (compare also Prop. 4.3 in [3]) that  $f = (Q)\text{-}\lim_I f_i$  means

$$(4.4) \quad f = (Q)\text{-}\liminf_I f_i \quad (\text{or } K_f = (Q)\text{-}\limsup_{I'} K_{f_i}) \quad \text{for every } I' \in \mathcal{J};$$

that is we have the stability of the  $(Q)\text{-}\liminf_I f_i$  (or  $(Q)\text{-}\limsup_{I'} K_{f_i}$ ) under the passages to the cofinal subsets of  $I$ . For the relation to the  $G$ -convergence of Marcellini see Remark 7.1 in [3].

We start with the following characterization which may be proved using Prop. 3.1 in a similar way as Theorem 5.1 in [3].

**PROPOSITION 4.1.** *Given a net  $\{f_i\}_I$  in  $\Gamma(F)$  and  $f \in \Gamma_0(F)$  the condition  $f = (Q)\text{-}\lim_I f_i$  can be characterized by the following two conditions:*

$$(i) \quad [(Q)\text{-}\limsup_I f_i](x) < +\infty \Rightarrow x \in \text{dom } f,$$

(ii) *For every  $I'$  cofinal in  $I$ ,  $f$  is the greatest function in  $\Gamma(F)$  with the property: for every  $\bar{x} \in \text{dom } f$ ,  $\varepsilon > 0$  there exist  $y \in G$ ,  $\varrho \in \mathbf{R}$  and a residual subset  $I''$  in  $I'$  such that*

$$\begin{aligned} f_i(x) &\geq \varrho + \langle x, y \rangle, & x \in F, & \quad i \in I'', \\ f(x) &\geq \varrho + \langle x, y \rangle, & x \in F, & \quad \text{and } f(\bar{x}) \leq \varrho + \langle \bar{x}, y \rangle + \varepsilon. \end{aligned}$$

From Prop. 4.1 easily follows

**PROPOSITION 4.2.** *Given a net  $\{f_i\}_I$  in  $\Gamma(F)$  a function  $f \in \Gamma_0(F)$  and an affine function  $g_{y\alpha}: F \ni x \mapsto \langle x, y \rangle + \alpha \in \mathbf{R}$ , we have*

$$f = (Q)\text{-}\lim_I f_i \Leftrightarrow f + g_{y\alpha} = (Q)\text{-}\lim_I (f_i + g_{y\alpha}).$$

**Remark 4.1.** Observe that under the lattice operations given by (4.3) and the order relation defined as usually by

$$f_1 \leq f_2 \quad \text{iff } f_1(x) \leq f_2(x) \quad \text{for every } x \in F,$$

the class

$$(4.5) \quad C(\alpha, x_0, M) = \{f \in \Gamma(F) \mid \alpha \leq f, f(x_0) \leq M\},$$

where  $\alpha \in \Gamma_0(F)$ ,  $x_0 \in F$ ,  $M \in \mathbf{R}$  are fixed and  $M \geq \alpha(x_0)$ , is complete in the sense that if  $f_i \in C(\alpha, x_0, M)$  then for any set of indices  $\bigvee f_i, \bigwedge f_i \in C(\alpha, x_0, M)$ . Moreover, it is

also closed with respect to the  $(Q)$ -convergence; i.e. if  $\{f_i\}_I \subset C(\alpha, x_0, M)$  is  $(Q)$ -convergent, then  $(Q)\text{-}\lim_I f_i \in C(\alpha, x_0, M)$ , and therefore  $(Q)\text{-}\lim_I f_i$  cannot be empty.

Remark 4.1 together with Theorem 3.1 immediately yield

**PROPOSITION 4.3.** *From any net  $\{f_i\}_I \subset C(\alpha, x_0, M)$  one can choose a cofinal subnet  $\{f_{\pi_n}\}_N$  which is  $(Q)$ -convergent to a function  $f \in C(\alpha, x_0, M)$ .*

Remark 4.2. Prop. 4.3 generalizes the result of Marcellini (see Teorema 2.4 in [2]), compare also Remark 7.1 in [3]) which was obtained by a different method in the case of a sequence  $\{f_n\}_N \subset C(\alpha, x_0, M) \subset \Gamma_0(F)$ , where  $F = V$  is a reflexive and separable Banach space and the function  $\alpha \in \Gamma_0(F)$  satisfies the following growth condition:

(G.C.) *All the level sets  $L_{y\beta}(\alpha) = \{x \in F \mid \alpha(x) - \langle x, y \rangle \leq \beta\}$ ,  $y \in G = V'$ ,  $\beta \in \mathbf{R}$  are bounded.*

Moreover, notice that in Prop. 4.3 neither reflexivity nor separability of the space  $F$  was needed, while the assumption  $\{f_i\}_I \subset C(\alpha, x_0, M)$  (with arbitrary  $\alpha \in \Gamma_0(F)$ ) was used only to assure the  $(Q)\text{-}\lim_{I'} f_{\pi_n}$  would be non-empty. Also Prop. 4.2 (in more general setting from one hand but only in the above class  $C(\alpha, x_0, M)$  from the other hand) was obtained by Marcellini in [7].

Using the anti-isomorphism  $\psi: \Gamma(F) \ni f \rightarrow f^* \in \Gamma(G)$  (see Section 2.5 in [3]), where the conjugate function  $f^*$  is defined by the formula

$$f^*(y) = \sup_{x \in F} [\langle x, y \rangle - f(x)], \quad y \in G,$$

we arrive (see Prop. 7.5 in [3]) at the following characterization.

**PROPOSITION 4.4.** *Given a net  $\{f_i\}_I$  in  $\Gamma(F)$  with totally ordered  $I$ , we have*

$$f = (Q)\text{-}\lim_I f_i \Leftrightarrow f^* = \bar{\Gamma}[\limsup_{I'} f_i^*] \quad \text{for every } I' \in \mathcal{J}$$

provided

$$(4.6) \quad (\limsup_{I'} f_i^*)(y) > -\infty, \quad \text{for every } y \in G \text{ and } I' \in \mathcal{J}.$$

Above  $\bar{\Gamma}[f]$  denotes the  $\sigma(F, G)$ -lower semi-continuous regularization of the function  $f$  (see Section 2.8 in [3]).

Now, for a fixed  $G_1 \subset G$  let us denote by  $\Gamma_1(F, G_1)$  a subclass of the class  $\Gamma(F)$  such that for every net  $\{f_i\}_I \subset \Gamma_1(F, G_1)$  with totally ordered  $I$  the conditions (4.6) and

$$(4.7) \quad \bar{\Gamma}[\limsup_{I'} f_i^*](y) = (\limsup_{I'} f_i^*)(y), \quad y \in G_1, I' \in \mathcal{J}$$

are fulfilled.

Remark 4.3 Observe (compare Remark 5.5 and 5.6 in [3]) that if the spaces  $F$  and  $G$  in duality are supplied, respectively, with the weak  $\sigma(F, G)$  and a  $\tau$  topology such that  $(G, \tau)$  is barrelled, then the class  $C(\alpha, x_0, M)$  defined by (4.5) with  $\alpha$  satisfying the relaxed growth condition:

(R. G. C.). The level sets  $L_{y\beta}(\alpha)$ ,  $y \in G_1$ ,  $\beta \in R$  are  $\sigma(F, G)$ -compact, where  $G_1$  is a  $\tau$ -open subset of  $G$ , provides an example of such a class  $\Gamma_1(F, G_1)$ .

Indeed, in the described case the topology  $\tau$  must coincide with the Mackey topology  $\tau(G, F)$  and since by Moreau-Rockafellar theorem (see [8] and [9])  $\alpha^*$  is finite and  $\tau(G, F)$ -continuous over  $G_1$ , so is the convex function  $\limsup_I f_i^*$  (for  $\langle x_0, y \rangle - M \leq f_i^*(y) \leq \alpha^*(y)$ ). Hence, condition (4.7) holds, (4.6) being obvious. If  $F = V$  is a reflexive and separable Banach space and  $G_1 = G = V'$  denotes its topological dual, then our class  $C(\alpha, x_0, M)$  coincides with that (as in Remark 4.2) considered by Marcellini in [6], [7], and [2].

From the above remark and from Prop. 4.4 we obtain at once

PROPOSITION 4.5. Given a net  $\{f_i\}_I \subset \Gamma_1(F, G_1)$  with totally ordered  $I$ , we have

$$f = (Q)\text{-}\lim_I f_i \Rightarrow f^*(y) = \lim_I f_i^*(y) \quad \text{for every } y \in G_1$$

and the sign " $\Rightarrow$ " may be replaced by " $\Leftrightarrow$ " provided  $G_1 = G$ .

Hence, owing to the fact that lower semi-continuous function attains its infimum over a compact set, we get

COROLLARY 4.1. If a net  $\{f_i\}_I$  with totally ordered  $I$  is contained in the class  $C(\alpha, x_0, M)$  as in Remark 4.3, then

$$f = (Q)\text{-}\lim_I f_i \Rightarrow \min_{x \in F} [f(x) - \langle x, y \rangle] = \lim_I \min_{x \in F} [f_i(x) - \langle x, y \rangle]$$

for every  $y \in G_1$  and we have " $\Leftrightarrow$ " provided  $G_1 = G$ .

Denoting by  $x_i(y)$  an element of  $F$  which realizes the  $\min_{x \in F} [f_i(x) - \langle x, y \rangle]$  (if such an element exists it need not be unique), we have

PROPOSITION 4.6. Under the assumption of Corollary 4.1 if  $f = (Q)\text{-}\lim_I f_i$ , then the elements  $x_i(y)$ ,  $i \in I$ ,  $y \in G_1$  and  $x(y)$  are well defined and fulfill the conditions:

$$(4.8) \quad \text{If } x_i(y) \mapsto x_\omega \text{ in } \sigma(F, G)\text{-topology, then } x_\omega = x(y)$$

realizes the  $\min_{x \in F} [f(x) - \langle x, y \rangle]$ ,

(4.9) If  $x(y)$  is unique (for instance if  $f$  is strictly convex), then  $x_i(y) \mapsto x(y)$  in  $\sigma(F, G)$ -topology.

Proof. Let us fix an element  $y \in G_1$ . All the elements  $x_i(y)$ ,  $i \in I$  and  $x(y)$  are well defined due to Corollary 4.1. From the assumption  $f = (Q)\text{-}\lim_I f_i$ , by Prop. 4.2 and formulae (4.1) (4.2), we get:

$$f(\cdot) - \langle \cdot, y \rangle = (Q)\text{-}\lim_I [f_i(\cdot) - \langle \cdot, y \rangle] = \bigvee_{\lambda \in I} \bigwedge_{i \succ \lambda} [f_i(\cdot) - \langle \cdot, y \rangle].$$

Since

$$\bigwedge_{i \succ \lambda} [f_i(\cdot) - \langle \cdot, y \rangle] \leq f_i(\cdot) - \langle \cdot, y \rangle, \quad \text{for any fixed } \lambda \in I, i \succ \lambda,$$

