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Approximation and decomposition theorems for the algebras of analytic functions in strictly pseudoconvex domains

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Abstract. In this paper we prove that Henkin's approximation theorem holds for the algebras $A^k(D)$ in strictly pseudoconvex domains with sufficiently smooth boundaries. We prove also the existence of the decomposition operators for the algebras $A^k(D)$ and $H^{\infty,k}(D)$ in strictly pseudoconvex domains in C^n.

0. Notations. We denote by C^n the complex n-dimensional space. The coordinates of the element z of C^n will be denoted by z_i, $i = 1, \dots, n$.

As usually, given $x \in C^n$ and $r>0$, $B(x, r)$ is a ball $\{z \in C^n: |z-x|<r\}$, $|\ |$ being the Euclidean norm in C^n.

We use the following notations for different spaces of analytic functions in a domain $D \subset C^n$:

$O(D)$ is the space of all functions holomorphic in D.

For a closed set $K \subset C^n$, $O(K)$ denotes the space of (germs of) functions which are holomorphic in some neighborhood of K.

For any non-negative integer k, $A^k(D)$ is the algebra of all functions, which are holomorphic in D and continuous with the derivatives of order $\leqslant k$ in $\bar{D}$, the closure of D in C^n. If $\alpha = (\alpha_1, \dots, \alpha_n) \in Z^n_+$ is an arbitrary n-tuple of non-negative integers, we set $|\alpha| = \alpha_1 + \dots + \alpha_n$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}.$$

Equipped with the norm

$$\|f\|_{D,k} = \sum_{|\alpha| \leqslant k} \sup_D |D^\alpha f|, \quad f \in A^k(D),$$

$A^k(D)$ becomes a Banach algebra. We will write $A(D)$ instead of $A^0(D)$ and $\|\ \|_D$ instead of $\|\ \|_{D,0}$.

As usually, we will denote by $H^\infty(D)$ the Banach algebra of all functions holomorphic and bounded in D, with the sup-norm.

Let D be a bounded domain in C^n. We say that D is strictly pseudoconvex with $\mathscr{C}^p$ boundary, $p \geqslant 2$, if there exists a real-valued function ϱ of class $\mathscr{C}^p$, defined in a neighborhood Ω of $\bar{D}$ and satisfying the following properties:

(i) ϱ is strictly plurisubharmonic in a neighborhood of ∂D;

(ii) for each $z \in \partial D$, $\operatorname{grad} \varrho(z) \neq 0$;

(iii) $D = \{z \in \Omega\colon \varrho(z) < 0\}$.

The function ϱ is called a defining function for D. We put $D_\eta = \{z \in \Omega\colon \varrho(z) < \eta\}$. If η is sufficiently close to zero, the domain D_η is also strictly pseudoconvex with $\mathscr{C}^p$ boundary.

If in the condition (i) we require ϱ to be only plurisubharmonic, we call D a weakly pseudoconvex domain with $\mathscr{C}^p$ boundary.

1. Introduction. This paper consists of two parts. In the first we consider the questions of approximation by functions holomorphic in a neighborhood of the closure of a strictly pseudoconvex domain. We prove the following generalization of Henkin's approximation theorem [5, p. 631]:

THEOREM 1. *Let D be a strictly pseudoconvex domain in C^n with $\mathscr{C}^p$ boundary. Let k be a non-negative integer such that $p \geqslant k+4$. Then the functions holomorphic in a neighborhood of $\bar{D}$ are dense in the algebra $A^k(D)$ (with $\| \ \|_{D,k}$-norm).*

In the case $k = 0$ the theorem was proved by Henkin [5] and independently by Kerzman [10] and Lieb [11]. Henkin's proof requires a weaker assumption on the differentiability of the boundary — it is sufficient to assume that D has a $\mathscr{C}^3$ boundary. Actually only $\mathscr{C}^2$ regularity of ∂D is needed, as was shown by Fornaess in [3].

At the end of the first part of the paper we give some applications of Theorem 1.

In the second part we study the existence of decomposition operators in different algebras of analytic functions in strictly pseudoconvex domains Let D be a domain in C^n and let $s = (s_1, \ldots, s_n)$ be a given point of D. Suppose that A is a subalgebra of functions holomorphic in D such that the coordinate functions $z_1, \ldots, z_n$ belong to A.

Definition. Any continuous linear operator

$$P_s\colon A \ni f \to \big((P_s f)_1, \ldots, (P_s f)_n\big) \in A^n$$

is called *a decomposition operator for the algebra A at a point $s \in D$* if for every $f \in A$ and for every $z \in D$,

$$f(z) = f(s) + (z_1 - s_1)(P_s f)_1(z) + \ldots + (z_n - s_n)(P_s f)_n(z). \tag{1.1}$$

If there exists a decomposition operator for A at each point of D, we say that A has a decomposition property.

Denote $f_i(z, s) = (P_s f)_i(z)$, $s, z \in \bar{D}$, $i = 1, \ldots, n$. We may then write (1.1) in the form

$$f(z) = f(s) + (z_1 - s_1) f_1(z, s) + \ldots + (z_n - s_n) f_n(z, s). \tag{1.1'}$$

Henkin [7] and Øvrelid [12] independently proved that a decomposition property holds for the algebra $A(D)$, D being a strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^2$ boundary. Moreover, Ahern and Schneider [16] improved this result by showing that the components f_j in (1.1)′ depend holomorphically on s, as s varies over D. In this paper we prove the existence of a decomposition operator for the algebra $A^k(D)$ in strictly pseudoconvex domain with sufficiently smooth boundary:

THEOREM 2. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^p$ boundary. Let k be a non-negative integer such that $p \geqslant k+4$. Then the algebra $A^k(D)$ has a decomposition property.*

Our proof follows that of [7] and we do not obtain the holomorphic dependence of the components in (1.1)′ on the second variable.

Let $H^{\infty,k}(D)$ be the algebra of all functions holomorphic in D such that their derivatives of order k are bounded in D. (Here k is a non-negative integer). $H^{\infty,k}(D)$ is a Banach algebra with $\|\ \|_{D,k}$-norm. In particular, $H^{\infty,0}(D) = H^{\infty}(D)$. We prove the following result on the decomposition property of $H^{\infty,k}(D)$:

THEOREM 3. *Let D, p and k be as in Theorem 2. For any $s \in D$, there exists a continuous operator*

$$R_s\colon\ H^{\infty,k}(D) \ni f \to (f_1, \ldots, f_n) \in (H^{\infty,k}(D))^n$$

such that for every $f \in H^{\infty,k}(D)$ and for every $z \in D$

$$f(z) = f(s)+(z_1-s_1)f_1(z)+\ldots+(z_n-s_n)f_n(z)\,.$$

We should stress that the operator R_s given by the above theorem is only continuous and need not be linear in general. The existence of the continuous and linear decomposition operator for $H^{\infty}(D)$ (with the holomorphic dependence of the components in (1.1)′ on the second variable) can be proved by a method similar to that of [16]. (In this case we need ∂D to be only of class $\mathscr{C}^3$).

For convenience, we recall here two results which are basic for our further investigations. We state them in a form suitable for our purposes.

THEOREM 4. (Henkin, [5, th. 1.1], [2, p. 86] and [6, p. 303]). *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^p$ boundary, and let ϱ be a defining function for D in a neighborhood Ω of $\bar{D}$. For any δ, $\varepsilon > 0$ define*

$$V_\delta = \{z \in \Omega\colon\ |\varrho(z)| < \delta\}\,, \qquad U_{\varepsilon,\delta} = \{(z, \xi) \in D_\delta \times V_\delta\colon\ |\xi - z| < \varepsilon\}\,.$$

There exist positive numbers γ, δ and ε, and the functions $\Phi(z, \xi) \in \mathscr{C}^{p-1}(D_\delta \times V_\delta)$, holomorphic with respect to z in D_δ, and $F(z, \xi)$, $G(z, \xi) \in \mathscr{C}^{p-1}(U_{\varepsilon,\delta})$, holomorphic with respect to z, such that

(i) *$\Phi = F \cdot G$ in $U_{\varepsilon,\delta}$, $F(\xi, \xi) = 0$, $|G(z, \xi)| > \gamma$ in $U_{\varepsilon,\delta}$, and $\Phi(z, \xi) > \gamma$ outside $U_{\varepsilon,\delta}$;*
(ii) *$2\operatorname{Re} F(z, \xi) \geqslant \varrho(\xi) - \varrho(z) + \gamma|\xi - z|^2$ in $U_{\varepsilon,\delta}$;*

(iii) *There exist the functions* $K_i(z,\xi)\in\mathscr{C}^{p-2}(D_\delta\times V_\delta)$, *holomorphic with respect to* z *in* D_δ, $i=1,\dots,n$, *such that if*

$$C(z,\xi)=\sum_{i=1}^{n}\frac{K_i(z,\xi)}{\Phi^n(z,\xi)}d\bar\xi_1\wedge\dots\wedge d\bar\xi_{i-1}\wedge d\bar\xi_{i+1}\wedge\dots\wedge d\bar\xi_n\wedge d\xi_1\wedge\dots\wedge d\xi_n,$$

then for every $u\in\mathscr{C}^1(\bar D)$ *and for every* $z\in D$,

$$u(z)=\int_{\partial D}u(\xi)C(z,\xi)+T_D(\bar\partial u)(z),$$

where T_D *is the integral operator which yields Henkin's solution of the* $\bar\partial$-*problem for the* $\bar\partial$-*closed* $\mathscr{C}^\infty(0,1)$-*forms in* $\bar D$ (see [6, p. 303] and [15, p. 165]);

(iv) *If* η *is close enough to zero, then for every* $u\in\mathscr{C}^1(\bar D)$ *and every* $z\in D_\eta$,

$$u(z)=\int_{\partial D}u(\xi)C(z,\xi)+T_{D_\eta}(\bar\partial u)(z)$$

with the same kernel $C(z,\xi)$. (*We recall that* $D_\eta=\{z\in\Omega:\ \varrho(z)<\eta\}$).

In (iii) and (iv) above we keep the notations of [15, p. 165].

THEOREM 5. (Siu [15]). *Let* D *be a strictly pseudoconvex domain in* $\mathbf{C}^n$ *with* $\mathscr{C}^p$ *boundary and let* p, k *be non-negative integers,* $p\geqslant k+4$. *For any* $(0,1)$-*form* f *with* $\mathscr{C}^\infty$ *coefficients in* D, *which satisfies the equation* $\bar\partial f=0$, *there exists* $u\in\mathscr{C}^\infty(D)$ *such that* $\bar\partial u=f$, *and*

$$\|u\|_{D,K}\leqslant c\cdot\|f\|_{D,k},$$

$c=c(D)$ *being a constant independent of* f.

Moreover, c *is independent of small perturbations of* D, *i.e. given a domain* D, *there exists* $c>0$ *such that* $c=c(D_\eta)$ *for any* η *close enough to zero.* (*If* $f=\sum_{i=1}^{n}f_i d\bar z_i$ *is a* $(0,1)$-*form with* $\mathscr{C}^k$ *coefficients in* D, *we put*

$$\|f\|_{D,k}=\sum_{i=1}^{n}\Big(\sum_{|\alpha|+|\beta|\leqslant k}\sup_D|D^\alpha\bar D^\beta f_i|\Big),$$

where $\alpha,\ \beta\in\mathbf{Z}^n_+$, *and*

$$\bar D^\beta=\frac{\partial^{|\beta|}}{\partial\bar z_1^{\beta_1}\dots\partial\bar z_n^{\beta_n}}.$$

2. The approximation theorem and applications. We give here a proof of the approximation theorem (Theorem 1).

We begin with the following lemma, which will be used also in proofs of the decomposition theorems:

LEMMA 6. *Let* D *be a strictly pseudoconvex domain in* $\mathbf{C}^n$ *with* $\mathscr{C}^p$ *boundary. Let* k *be a non-negative integer such that* $p\geqslant k+4$. *Suppose that* g *is a* $\mathscr{C}^\infty$ *function defined in* *a neighborhood Ω of $\bar{D}$. Given $f \in A^k(D)$, define the function f_g by*

(2.1) $$f_g(z) = \int_{\partial D} f(\xi)g(\xi)C(z,\xi), \quad z \in D.$$

Then $f \in A^k(D)$ and satisfies the estimate

(2.2) $$\|f_g\|_{D,k} \leqslant c\cdot\|f\|_{D,k}\|g\|_{D,k+1},$$

c being a constant independent of f.

For the case $k = 0$, the lemma was proved by Henkin [5, Lemma 4.3] in a more general setting, which is not necessary for our purposes. Our proof is based on the results of Siu [15].

Proof of Lemma 6. We may assume $k>0$. For every $z \in D$,

(2.3) $$f(z)g(z) = \int_{\partial D} f(\xi)g(\xi)C(z,\xi) + T_D(\bar\partial(f\cdot g))(z) = f_g(z) + T_D(f\cdot\bar\partial g)(z)$$

by Theorem 4, (iii). Let ϱ be a defining function for D and let

$$D_m = \{z \in D\colon \varrho(z) < -1/m\}, \quad m = 1, 2, \ldots.$$

Then for m sufficiently large,

(2.4) $$f(z)\cdot g(z) = \int_{\partial D_m} f(\xi)g(\xi)C(z,\xi) + T_{D_m}(f\cdot\bar\partial g)(z) = (f_g)_m(z) + T_{D_m}(f\cdot\bar\partial g)(z), \quad z \in D_m,$$

the integral kernel $C(z, \xi)$ being the same as for D, in virtue of Theorem 4, (iv). Since the sequence $\{(f_g)_m\}$ converges to f_g uniformly on compact subsets of D, we conclude from (2.3) and (2.4) that $T_{D_m}(f\cdot\bar\partial g)$ converges to $T_D(f\cdot\bar\partial g)$ uniformly on compact subsets of D. The functions $T_{D_m}(f\cdot\bar\partial g)$ are bounded on their domains of definition by the same constant ([6], [10]), and the differences $T_{D_m}(f\cdot\bar\partial g) - T_{D_l}(f\cdot\bar\partial g)$ are holomorphic wherever they are defined. Therefore, by a standard use of Montel's theorem, Cantor diagonal process and Weierstrass theorem, we conclude (as in [15]) that there exists a subsequence $\{T_j\}$ of the sequence $\{T_{D_m}(f\cdot\bar\partial g)\}$ such that for any $\alpha \in Z^n_+$, the sequence $\{D^\alpha T_j\}$ converges uniformly on compact subsets of D to some function h_α. Therefore $h_\alpha = D^\alpha T_D(f\cdot\bar\partial g)$. By [15, p. 175], for any $\alpha \in Z^n_+$, with $|\alpha| \leqslant k$, the function h_α extends continuously to all of $\bar D$ and

(2.5) $$\|h_\alpha\|_D \leqslant c\cdot\|f\cdot\bar\partial g\|_{D,k}$$

for some c independent of f. Since

(2.6) $$D^\alpha f_g = D^\alpha(f\cdot g) - D^\alpha T_D(f\cdot\bar\partial g) = D^\alpha(f\cdot g) - h_\alpha,$$

it follows that $D^\alpha f_g$ extends continuously up to the boundary of D, for $|\alpha| \leqslant k$. Thus $f_g \in A^k(D)$, and the estimate (2.2) follows from (2.5) and (2.6).

Proof of Theorem 1. Fix $\varepsilon>0$. For any $\xi \in \partial D$ there exists a neighborhood U_ξ of ξ in C^n and a number $\delta_\xi>0$ such that

(2.7) $$U_\xi \cap \bar D \subset D + \delta\cdot n(\xi)$$

for $0<\delta<\delta_\xi$. (Here $n(\xi)$ denotes the exterior unit normal to ∂D at ξ.) Choose a finite number of points $\xi_1, \ldots, \xi_N \in \partial D$ and the positive numbers $r_1, \ldots, r_N$ such that $U_{\xi_i} = B(\xi_i, r_i)$ satisfy (2.7), $i = 1, \ldots, N$ and $\partial D \subset \bigcup_{i=1}^{N} B(\xi_i, r_i/2)$. Let g_i, $i = 1, \ldots, N$ be a $\mathscr{C}^\infty$ partition of unity in a neighborhood of ∂D, subordinate to the covering $(B(\xi_i, r_i/2))$ $i = 1, \ldots, N$. Given $f \in A^k(D)$, we then have

$$f(z) = \sum_{i=1}^{N} f_{g_i}(z), \quad z \in \bar{D} \tag{2.8}$$

with f_{g_i} defined by (2.1). Since $g_i \equiv 0$ outside $B(\xi_i, r_i/2)$, we have

$$f_{g_i}(z) = \int_{\partial D \cap B(\xi_i, r_i/2)} f(\xi) g_i(\xi) C(z, \xi) .$$

By (i) and (ii) of Theorem 4 and the definition of $C(z, \xi)$ we conclude that each f_{g_i} is in fact analytic in some neighborhood of $\partial D \setminus B(\xi_i, \frac{2}{3} r_i)$. Combining this with (2.7) we conclude that for each $i = 1, \ldots, N$ there exists δ_i, $0<\delta_i \leqslant \delta_{\xi_i}$ so small that the function $f_i(z) = f_{g_i}(z-\delta_i \cdot n(\xi_i))$ is holomorphic in a neighborhood of $\bar{D}$ and

$$\|f_{g_i} - f_i\|_{D,k} \leqslant \varepsilon/N . \tag{2.9}$$

It follows that $\sum_{i=1}^{N} f_i \in O(D)$ and, by (2.8) and (2.9),

$$\|f - \sum_{i=1}^{N} f_i\|_{D,k} < \varepsilon .$$

Since ε was arbitrary, the proof is concluded.

We give also an alternative proof of Theorem 1, based on a method set forth by Kerzman [10]. Since the proof follows the same line as that of [10, th. 1.4.1], we give here only the necessary modifications.

Proof of Theorem 1 by Kerzman's method. We keep the notation of [10]. Let D be a strictly pseudoconvex domain in C^n with $\mathscr{C}^p$ boundary. Consider a function $u \in A^k(D)$. The construction of the covering (V_i^δ), $i = 0, \ldots, N$ of regions D^δ and of the holomorphic cocycle (v_{ij}^δ), $v_{ij}^\delta = v_i^\delta - v_j^\delta \in O(V_i^\delta \cap V_j^\delta)$, $0<\delta<\delta_0$, $i, j = 0, \ldots, N$, is similar to that given in [10]. Note that under the assumption of Theorem 1 the functions v_{ij}^δ are in $A^k(V_i^\delta \cap V_j^\delta)$.

In order to continue the proof we have to replace Claim 2 of [10] by the following

CLAIM. *There are functions $h_i^\delta \in A^k(V_i^\delta)$ such that $h_i^\delta - h_j^\delta = v_{ij}^\delta$ on $V_i^\delta \cap V_j^\delta$, and which satisfy*

$$\sup_i \|h_i^\delta\|_{V_i^\delta,k} \leqslant c \sup_{i,j} \|v_{ij}^\delta\|_{V_i^\delta \cap V_j^\delta,k}, \quad i, j = 0, \ldots, N, \tag{*}$$

the constant c being independent of u and δ.

Proof of the Claim. In the same way as in [10], we define the open covering $(U_i^\delta \cap D^\delta)_{i=0,\ldots,N}$ of D^δ and the $\mathscr{C}^\infty$ partition of unity (Φ_i^δ) $i = 0, \ldots, N$, in a neighborhood of $\bar{D}^\delta$, subordinate to this covering. It follows that

$$\sup_j \|\bar{\partial} \Phi_j^\delta\|_{D^\delta,k} \leqslant A \tag{2.10}$$

for some constant A independent of δ, provided that δ is small enough. Then $f^\delta = \sum_{j=0}^{N} v_{ij}^\delta \cdot \bar\partial \Phi_j^\delta$, $i = 0, \dots, N$, is a well-defined $\mathscr{C}^\infty$ (0, 1)-form in D^δ such that $\bar\partial f^\delta = 0$, and the coefficients of f^δ are in $\mathscr{C}^k(\bar D^\delta)$. Using (2.10) it is easy to show that

$$(2.11) \qquad \|f^\delta\|_{D^\delta,k} \leqslant c \sup_{i,j} \|v_{ij}^\delta\|_{V_i^\delta \cap V_j^\delta,k},$$

where c is a constant independent of δ for δ small enough.

Now we use Theorem 5 instead of [10, th. 1.2.1]. It follows that there exists $\tilde u$ which is $\mathscr{C}^\infty$ in D^δ and $\mathscr{C}^k$ in $\bar D^\delta$, such that $\bar\partial \tilde u = f^\delta$, and which satisfies

$$(2.12) \qquad \|\tilde u\|_{D^\delta,k} \leqslant c \|f^\delta\|_{D^\delta,k},$$

the constant c being independent of u and δ, in virtue of the perturbation result of Theorem 5.

Define

$$(2.13) \qquad w_i^\delta = \sum_{j=0}^{N} v_{ij}^\delta \Phi_j^\delta \colon V_i^\delta \to \mathbf{C}, \quad i = 0, \dots, N,$$

and let $h_i^\delta = w_i^\delta - \tilde u$. Reasoning as in [10] (with the obvious modifications) we conclude that h_i^δ are in $A^k(V_i^\delta)$, $i = 0, \dots, N$. The assertion of the Claim now follows from (2.11), (2.12) and (2.13).

Having proved the Claim, we return to the proof of Theorem 1. The function $v^\delta = v_i^\delta - h_i^\delta = v_j^\delta - h_j^\delta$ is well-defined and holomorphic in all of D^δ, $0 < \delta < \delta_0$. Since D^δ has a Runge property with respect to D^{δ_0} (this is a consequence of Theorems 4.3.4 and 4.3.3 of [9]) it follows that v^δ can be uniformly approximated on $\bar D^{\delta/2}$ by functions which are holomorphic in D^{δ_0}. We conclude from [9, Theorem 2.2.3] that v^δ can be approximated in $\| \ \|_{D,k}$-norm by functions holomorphic in D^{δ_0}. Since $\|v_{ij}^\delta\|_{V_i^\delta \cap V_j^\delta,k} \to 0$ as $\delta \to 0$, (*) implies that also $\|h_0\|_{V_0^\delta,k} \to 0$, $\delta \to 0$. But $V_0^\delta = G$, $v_0^\delta = u$ and $v^\delta = v_0^\delta - h_0^\delta = u - h_0^\delta$ on G. Hence $\|u - v^\delta\|_{D,k} = \|h_0^\delta\|_{D,k} \to 0$ as $\delta \to 0$, and since v^δ is approximable in $\| \ \|_{D,k}$-norm by functions holomorphic in D^{δ_0}, so is u. The proof of Theorem 1 is thus completed.

Note. Theorem 1 is also valid for $p = k = \infty$. (In this case the algebra $A^\infty(D)$ of functions holomorphic in D and continuous in $\bar D$ with all their derivatives is considered with the topology of uniform convergence of all derivatives on $\bar D$).

Henkin's approximation theorem (Theorem 1 with $k = 0$) is an essential step in the proofs of a number of results. An application of Theorem 1 with different values of k allows to extend those results to a more general setting. We give here a few examples.

One of them is the following improvement of a theorem by Sibony and Wermer [14]:

THEOREM 7. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^{k+5}$ boundary. Suppose that $f_1, \dots, f_m \in A^{k+5}(D)$ satisfy the following conditions*:

(i) *The f_i separate points of $\bar D$.*

(ii) *The matrix $[\partial f_i / \partial z_j]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ has rank n for all $z \in \bar D$.*

(iii) $K = \{(f_1(z), \dots, f_m(z)) \colon z \in \bar{D}\}$ *is polynomially convex in* C^m. *Then* $f_1, \dots, f_m$ *are a set of generators for* $A^k(D)$.

In the case $k = 0$ the theorem was proved by Sibony and Wermer in [14] for strictly pseudoconvex domains with $\mathscr{C}^4$ boundary, and Rossi and Taylor showed in [13] that it is enough to assume that ∂D is of class $\mathscr{C}^3$.

In order to prove Theorem 7 it is sufficient to note that the domain E in Theorem 4.8 of [13] has $\mathscr{C}^{k+4}$ boundary and then use Theorem 1 and the following lemma which is an easy consequence of Oka–Weyl approximation theorem:

LEMMA 8. *Let* K *be a polynomially convex set in* C^n. *Suppose that the functions of the algebra* $O(K)$ *are dense in* $A^k(K)$ *(with the* $\| \ \|_{K,k}$*-norm). Then the polynomials are dense in* $A^k(K)$ *in the* $\| \ \|_{K,k}$*-norm.*

Next we give extensions of the approximation theorems for weakly pseudoconvex domains with sufficiently small number of weakly pseudoconvex boundary points:

THEOREM 9. (Fornaess and Nagel [4]). *Let* D *be a weakly pseudoconvex domain in* C^n *with* $\mathscr{C}^p$ *boundary and let* k *be an integer such that* $p \geq k+4$. *Denote by* M *the set of weakly pseudoconvex boundary points of* ∂D. *Suppose that there exists an open neighborhood* U *of* M *in* C^n *and a holomorphic mapping* $F\colon U \to C^n$ *such that* $\mathrm{Re}\langle F(\xi), n(\xi)\rangle > 0$ *for* $\xi \in \partial D \cap U$. *Then the functions holomorphic in a neighborhood of* $\bar{D}$ *are dense in the algebra* $A^k(D)$ *with the* $\| \ \|_{D,k}$*-norm. (Here* $n(\xi)$ *denotes the exterior unit normal to* ∂D *at* ξ*).*

THEOREM 10 (Fornaess and Nagel [4]). *Let* D, M, p *and* k *be as above. Suppose that* M *is a stratified union of totally real* $\mathscr{C}^\infty$ *manifolds* (see [4]). *Then the assertion of Theorem* 9 *holds.*

In particular, the assumptions of the above theorem are satisfied for any pseudoconvex domain with $\mathscr{C}^p$ boundary and with the finite number of points of weak pseudoconvexity in ∂D ([4]).

For the case $k = 0$ those results were obtained by Fornaess and Nagel in [4].

3. The decomposition theorems. In this part we give the proofs of decomposition theorems (Theorems 2 and 3).

Proof of Theorem 2. We include here the detailed proof for the case $k = 0$, since it differs from that of [16] and will be used also for the case $k \geq 1$.

We will verify that the construction similar to that given in [7] yields the decomposition operator for the algebra $A^k(D)$. In order to simplify notation, we assume that $s = 0$. Since ∂D is of class $\mathscr{C}^2$, for every $\xi_0 \in \partial D$ there exists a matrix $M = [m_{jk}]_{j,k=1,\dots,n}$, a ball $B(\xi_0, r)$ and a number $\varepsilon_0 > 0$ such that for each ε with $0 < \varepsilon \leq \varepsilon_0$, there exists a neighborhood V_ε of $\partial D \cap B(\xi_0, r/2)$ which is relatively compact in $B(\xi_0, r)$ and such that for any $z \in V_\varepsilon$.

$$z - \varepsilon \cdot Mz \in D. \tag{3.1}$$

Choose a finite number of balls $B(\xi_i, \delta)$, $i = 1, \ldots, N$, of the above type such that $\partial D \subset \bigcup_{i=1}^{N} B(\xi_i, \delta/4)$. As in the proof of Theorem 1, given $f \in A^k(D)$, we can find the functions $g_i \in A^k(D)$, $i = 1, \ldots, N$ which are continuable holomorphically across $\partial D \setminus B(\xi_i, \delta/4)$ and which satisfy

$$f = \sum_{i=1}^{N} g_i \tag{3.2}$$

and

$$\sup_i \|g_i\|_{D,k} \leqslant c \cdot \|f\|_{D,k}, \tag{3.3}$$

the constant c being independent of f. In addition, g_i's can be chosen in such a way that for some neighborhood U_i of $\partial D \setminus B(\xi_i, \delta/2)$, g_i is holomorphic in $D \cup U_i$, $i = 1, \ldots, N$, and

$$\sup_i \|g_i\|_{D \cup U_i, k} \leqslant c \cdot \|f\|_{D,k} \tag{3.4}$$

for some constant c independent of f, $i = 1, \ldots, N$. We conclude from (3.2) and (3.3) that in order to prove the theorem, it is sufficient to show that for each $i = 1, \ldots, N$, there exist functions $g_1^{(i)}, \ldots, g_n^{(i)} \in A^k(D)$ such that

$$g_i(z) = g_i(0) + \sum_{j=1}^{n} z_j g_j^{(i)}(z), \qquad z \in \bar{D}$$

and

$$\|g_j^{(i)}\|_{D,k} \leqslant c \cdot \|f\|_{D,k}, \qquad j = 1, \ldots, n.$$

Fix i, $1 \leqslant i \leqslant N$, and denote $g = g_i$ and $\xi_0 = \xi_i$. Choose $\varepsilon_0 > 0$ and a matrix $M = [m_{ik}]_{i,k=1,\ldots,n}$ such that (3.1) holds for each $0 < \varepsilon < \varepsilon_0$ and $z \in V_\varepsilon$, where V_ε is a convenient neighborhood of $\partial D \cap B(\xi_0, \delta/2)$. Since $\partial D \subset U_i \cup V_\varepsilon$ for each ε, $0 < \varepsilon \leqslant \varepsilon_0$, we may assume that (after shrinking ε_0 if necessary) there exists a neighborhood U_ε of $\bar{D}$ (which depends on ε) such that the function

$$g_\varepsilon(z) = g(z - \varepsilon \cdot Mz) \tag{3.5}$$

is defined in U_ε, $0 < \varepsilon \leqslant \varepsilon_0$, and

$$\|g_\varepsilon\|_{U_\varepsilon, k} \leqslant c \cdot \|f\|_{D,k}. \tag{3.6}$$

In addition, we may choose U_ε in such a way that there exists a neighborhood U of $\partial D \setminus B(\xi_0, \delta/2)$ which is contained in each U_ε, $0 < \varepsilon \leqslant \varepsilon_0$.

We recall that the function g is defined by

$$g(z) = \int_{\partial D \cap K} f(\xi) \tilde{g}(\xi) C(z, \xi), \qquad z \in D$$

where $\tilde{g}$ is some $\mathscr{C}^\infty$ function in C^n and $K = \operatorname{supp} \tilde{g} \subset B(\xi_0, \delta/4)$. Thus

$$g_{\varepsilon_0}(z) = \int_{\partial D \cap K} f(\xi) g(\xi) C(z - \varepsilon_0 \cdot Mz, \xi), \qquad z \in U_{\varepsilon_0}.$$