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On an iterative functional inequality

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In this paper we shall deal with the inequality

(1) $$\psi^2(x) \leqslant G(x, \psi(x)),$$

where G is a given function, ψ is an unknown function and ψ^i denotes i-th iterate of the function ψ. Inequality (1) has not been studied so far. In our research we shall follow the trend initiated by D. Brydak [1], i.e. we shall consider inequality (1) under assumptions taken from the theory of the equation

(2) $$\varphi^2(x) = G(x, \varphi(x)).$$

Equation (2) was studied by M. Kuczma [3], M. K. Fort [2], P. E. Lush [5] and their results are to be found in [4] (ch. XIV, § 3)

Throughout the paper we shall assume conditions which are usually imposed for equation (2) (cf. [4], ch. XIV, § 3).

(H) I. $G\colon \Omega \to R$ is a function continuous in the domain

$$\Omega = \{(x, y)\colon\ x \in [0, b], b>0, 0\leqslant y \leqslant x\},$$

II. G is strictly increasing with respect to each variable,

III. $G(0, 0) = 0,\ G(b, b) = b$,

IV. $G(x, x) < x$ for $x \in (0, b)$,

V. There exists a function $\beta\colon [0, b] \to R$ such that

$$G(x, \beta(x)) = \beta(x) \quad \text{for } x \in [0, b]$$

and the inequality

(3) $$G(x, y) < y$$

holds in the set $\{(x, y)\colon 0<x<b, \beta(x)<y<x\}$.

It follows from these assumptions (cf. [4], ch. XIV, § 3) that the function β is continuous, strictly increasing in the interval $[0, b]$ and

(4) $$\beta(0) = 0, \quad \beta(b) = b,$$

(5) $$\beta(x) < x \quad \text{for} \quad x \in (0, b).$$

§ **1.** In this section we shall give some sufficient and necessary conditions for a continuous function ψ to be a solution of inequality (1). These results are analogous to the theorems contained in [6] and concerning the inequality

$$\psi^n(x) \leqslant g(x),$$

Definition 1. We denote by C_0^b the class of continuous functions ψ: $[0, b] \to [0, b]$ fulfilling the inequality

$$\psi(x) < x \quad \text{for } x \in (0, b). \tag{6}$$

Remark 1. If a function G: $[0, b]^2 \to R$ satisfies assumptions (H) then, as it is easily checked, every continuous solution of inequality (1) fulfills also inequality (6). Thus assumption (6) in the above definition is not restrictive.

THEOREM 1. *Let hypothesis* (H) *be fulfilled. If a function* $\psi \in C_0^b$ *fulfils the inequality*

$$\psi(x) \leqslant G(x, x) \quad \textit{for} \quad x \in [0, b], \tag{7}$$

then ψ *fulfills inequality* (1) *in* $[0, b]$.

Proof. It follows from (7) by virtue of (6) and (H, II), that

$$\psi^2(x) \leqslant G(\psi(x), \psi(x)) \leqslant G(x, \psi(x)) \quad \text{for} \quad x \in [0, b],$$

which ends the proof.

THEOREM 2. *Assume that hypothesis* (H) *is fulfilled and* $\psi \in C_0^b$. *If there exist continuous functions* p: $(0, b) \to (0, 1)$ *and* q: $(0, b) \to (1, \infty)$ *fulfilling in* $(0, b)$ *the following inequalities*

$$\psi(x) \leqslant p(x) G(x, \psi(x)) + (1 - p(x)) x, \tag{8}$$

$$p(x) G(x, \psi(x)) + (1 - p(x)) x \leqslant q(x) G(x, x), \tag{9}$$

$$q(\psi(x)) G(\psi(x), \psi(x)) \leqslant G(x, \psi(x)), \tag{10}$$

then ψ *fulfills inequality* (1) *in* $[0, b]$.

Proof. Putting $\psi(x)$ in place of x in (8) we have, by virtue of (9) and (10)

$$\psi^2(x) \leqslant p(\psi(x)) G(\psi(x), \psi^2(x)) + (1 - p(\psi(x))) \psi(x)$$
$$\leqslant q(\psi(x)) G(\psi(x), \psi(x)) \leqslant G(x, \psi(x)) \quad \text{for } x \in (0, b).$$

The obtained inequality

$$\psi^2(x) \leqslant G(x, \psi(x))$$

holds also at points $x = 0$ and $x = b$ because ψ and G are continuous in these points, which ends the proof.

Assumption (8) in theorem 2 is weaker than that (7) in theorem 1. The theorem 3, proved below, shows that the additional conditions (9) and (10) are, in a sense, necessary.

THEOREM 3. *Let hypothesis* (H) *be fulfilled. If a function* $\psi \in C_0^b$ *fulfills the inequality*

$$G(x, x) < \psi(x) \quad \text{for } x \in (0, b) \tag{11}$$

and it is strictly increasing solution of inequality (1) *in* $[0, b]$, *then there exist continuous functions* $p\colon (0, b) \to (0, 1)$ *fulfilling inequality* (8) *and* $q\colon (0, b) \to (1, \infty)$ *fulfilling inequalities* (9) *and* (10).

Proof. The function

$$p(x) = \frac{\psi(x) - x}{G(x, \psi(x)) - x}$$

is defined, continuous and positive in $(0, b)$ and it is obvious that p fulfills inequality (8). The inequality

$$p(x) < 1 \quad \text{for } x \in (0, b)$$

follows from (11) and (H, IV) (note that $\psi(x) < x$ in $(0, b)$).

Now we shall prove the part of the thesis concerning the function q. Let us put

$$r(x) := \frac{G(\psi^{-1}(x), x)}{\psi(x)} \quad \text{for } x \in (0, b) . \tag{12}$$

Then we have

$$r(x) \geq 1 \quad \text{for } x \in (0, b) .$$

Equality (12) is equivalent to the following

$$\frac{G(\psi^{-1}(x), x)}{G(x, x)} = \frac{r(x)}{G(x, x)} \left[\frac{\psi(x) - x}{G(x, \psi(x)) - x} G(x, \psi(x)) + x - \frac{\psi(x) - x}{G(x, \psi(x)) - x} x \right]. \tag{13}$$

Let us define

$$q(x) := \frac{G(\psi^{-1}(x), x)}{G(x, x)} = \frac{r(x)\psi(x)}{G(x, x)} \quad \text{for } x \in (0, b) . \tag{14}$$

Equality (13) implies, by virtue of the definition of the function p and (14), the following equality

$$q(x) = r(x) \frac{p(x) G(x, \psi(x)) + (1 - p(x))x}{G(x, x)} . \tag{15}$$

We shall prove that the function q fulfills (9) and (10). Since $r(x) \geq 1$ in $(0, b)$, then (15) implies the inequality

$$q(x) \geq \frac{p(x) G(x, \psi(x)) + (1 - p(x))x}{G(x, x)} ,$$

whence we have (9).

Putting $\psi(x)$ in place of x in (14) we obtain

$$q(\psi(x)) = \frac{G(x, \psi(x))}{G(\psi(x), \psi(x))} \quad \text{for } x \in (0, b)$$

e.g. (10) (with "=" in place of "≤").

The inequality $q(x) \geq 1$ in $(0, b)$ follows from the inequality $r(x) \geq 1$ in $(0, b)$ and from the second equality (14), which completes the proof.

§ **2.** Now we quote here (as lemma 1) some definition and facts due to M. Kuczma [4] (ch. XIV, § 3) which will be useful in the sequel.

Definition 2. Let Ψ denote the space of all continuous and strictly increasing in $[0, b]$ functions ψ fulfilling the relation

(16) $$\beta(x) \leqslant \psi(x) \leqslant x \quad \text{for } x \in [0, b],$$

with the metric

$$\varrho(\psi_1, \psi_2) := \sup_{[0,b]} |\psi_1(x), -\psi_2(x)| .$$

LEMMA 1. *A transformation* T: $\Psi \to \Psi$ *defined by the formula*

(17) $$T[\psi](x) = G(\psi^{-1}(x), x) \quad \textit{for } x \in [0, b],$$

is a continuous and one-to-one transformation of the space Ψ *into itself, order reversing* i.e.

(18) $$\psi_1 \leqslant \psi_2 \Rightarrow T[\psi_1] \geqslant T[\psi_2]$$

$$(\psi_1 \leqslant \psi_2 \Leftrightarrow \psi_1(x) \leqslant \psi_2(x) \quad \textit{for } x \ [0, b]) .$$

If hypothesis (H) *is fulfilled, then*

(19) $$\overline{T(\Psi)} \subset \Psi$$

(here $\bar{C}$ *denotes closure of the set* C *in the space of all continuous functions),*

(20) $$T^2(\Psi) \textit{ is an equicontinuous set of functions} .$$

Moreover, we have also

(21) $$T^0[\beta](x) := \beta(x) \leqslant T^2[\beta](x) = G(x, x) \leqslant \ldots$$
$$\leqslant T^3[\beta](x) \leqslant T^1[\beta](x) = x \quad \textit{for } x \in [0, b] .$$

LEMMA 2. *Assume that hypothesis* (H) *is fulfilled and function* $\psi \in \Psi$ *fulfills the inequality* (1) *in* $[0, b]$. *Let us put*

(22) $$\alpha_n(x) := T^n[\alpha](x) \quad \textit{for } x \in [0, b],\ n = 0, 1, \ldots$$

If the sequence α_{2n} is increasing, then there exist continuous functions $t, s \in \Psi$ such that

(23) $$t(x) = \lim_{n\to\infty} \alpha_{2n}(x) \quad \textit{for } x \in [0, b],$$

(24) $$s(x) = \lim_{n\to\infty} \alpha_{2n+1}(x) \quad \textit{for } x \in [0, b]$$

and

(25) $$t(x) \leqslant s(x) \quad \textit{for } x \in [0, b].$$

Proof. Putting $\alpha^{-1}(x)$ in place of x in (1) we have $\alpha_0(x) = \alpha(x) \leqslant G(\alpha^{-1}(x), x) = \alpha_1(x)$ for $x \in [0, b]$. Hence $\alpha_0 \leqslant \alpha_1$. Now (18) implies the following inequalities

$$\alpha_2 = T[\alpha_1] \leqslant T[\alpha_0] = \alpha_1,$$
$$\alpha_2 = T[\alpha_1] \leqslant T[\alpha_2] = \alpha_3$$

and in general (by virtue of monotonicity of the sequence α_{2n})

(26) $$\alpha_{2k} \leqslant \alpha_{2p+1} \quad \text{for } k = 0, 1, \ldots\ p = 0, 1, \ldots$$

Since the sequence α_{2n} is increasing, then the sequence α_{2n+1} is decreasing, whence, by virtue of (26), we have

(27) $$\alpha_0 \leqslant \alpha_2 \leqslant \alpha_4 \leqslant \ldots \leqslant \alpha_5 \leqslant \alpha_3 \leqslant \alpha_1 .$$

It follows from (27) that the sequences α_{2n} and α_{2n+1} are convergent, therefore the functions t and s exist, and that inequality (25) holds. Since the convergence of sequences α_{2n} and α_{2n+1} is uniform, then t and s are continuous. They belong to Ψ, by virtue of (19), what ends the proof.

THEOREM 4. *Let hypothesis* (H) *be fulfilled. If a function $\alpha \in \Psi$ is such a solution of* (1) *for which the sequence α_{2n} given by* (22) *is increasing, then each function $\psi \in \Psi$ fulfilling inequality*

(28) $$\psi(x) \leqslant t(x) \quad \textit{for } x \in [0, b],$$

where function t is given by (23), *fulfills also* (1) *in* $[0, b]$.

Proof. From continuity of transformation T we have

$$T[t] = T[\lim_{n\to\infty} \alpha_{2n}] = \lim_{n\to\infty} T[\alpha_{2n}] = \lim_{n\to\infty} \alpha_{2n+1} = s,$$
$$T[s] = T[\lim_{n\to\infty} \alpha_{2n+1}] = \lim_{n\to\infty} T[\alpha_{2n+1} = \lim_{n\to\infty} \alpha_{2n+2} = t.$$

Inequality (28) and (24) imply inequality $\psi^{-1} \geqslant s^{-1}$.
Hence by virtue of (H, II) and $T[s] = t$ we obtain

$$\psi^2(x) = \psi(\psi(x)) \leqslant t(\psi(x)) = T[s](\psi(x))$$
$$= G(s^{-1}(\psi(x), \psi(x)) \leqslant G(\psi^{-1}(\psi(x)), \psi(x)) = G(x, \psi(x))$$

what ends the proof.

Since it is easy to verify that the function β fulfills the assumptions of theorem 4, we have the following

Corollary 1. *If the function G fulfills hypothesis* (H), *then every function* $\psi \in \Psi$ *fulfilling inequality*

$$\psi(x) \leqslant \tau(x) \quad \textit{for } x \in [0, b],$$

where

$$\tau(x) := \lim_{n\to\infty} \beta_{2n}(x) \quad \textit{for } x \in [0, b],$$

$$\beta_n(x) := T^n[\beta](x) \quad \textit{for } x \in [0, b],\ n = 0, 1, \ldots$$

fulfills also inequality (1) *in* $[0, b]$.

A continuous solution φ of equation (2) in $[0, b]$ also fulfills the assumptions of theorem 4 (see [4], ch. XIV, § 3) and it is obvious that

$$\varphi_n := T^n[\varphi] = \varphi \quad \text{for } n = 0, 1, \ldots$$

This imply the following

Corollary 2. *Let the hypothesis* (H) *be fulfilled. If a function* φ *is continuous solution of equation* (2) *in* $[0, b]$ *and a* $\psi \in \Psi$ *fulfills the inequality*

$$\psi(x) \leqslant \varphi(x) \quad \textit{for } x \in [0, b],$$

then ψ *fulfills inequality* (1) *in* $[0, b]$.

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