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On turbulent statistical solutions of the heat equation

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1. Introduction. The notion of the statistical solution of the heat equation describes the phenomena of heat transfer with random initial temperature and heat sources (see [1]). It has its origin in Foias' paper [2]. In this paper we introduce the notion of turbulent solutions of the heat equation. We give the necessary and sufficient condition for the existence of such solutions.

2. Preliminaries. By L^2 we shall denote the Hilbert space of all square integrable real valued functions u defined on Ω — a bounded domain in R^n with boundary of class C^2. The norm of $u \in L^2$ will be denoted by $\|u\|$ and the scalar product of $u, v \in L^2$ by (u, v). We define the subspace H_0^1 of L^2 in the following way: H_0^1 is the closure of $C_0^\infty(\Omega)$ in H^1 where

$$H^1 = \left\{u \in L^2 : \frac{\partial u}{\partial x_k} \in L^2,\ k = 1, \ldots, n\right\}$$

(derivatives are taken in the sense of the theory of distributions) with norm

$$\|u\|_{H^1} = \left(\|u\|^2 + \sum_{k=1}^{n} \left\|\frac{\partial u}{\partial x_k}\right\|^2\right)^{\frac{1}{2}}.$$

The space H_0^1 is a Hilbert space with a scalar product

$$(u, v)_1 = \sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}\right) \quad \text{for } u, v \in H_0^1 .$$

We denote $\|u\|_1 = ((u, u)_1)^{\frac{1}{2}}$ for $u \in H_0^1$.

Definition 1. We call $F: L^2 \times L^2 \to R$ a *test functional* if

1) it is continuous,

2) for some constants c_1, c_2, c_3 we have

$$|F(u, f)| < c_1 + c_2 \|u\| + c_3 \|f\|$$

for all $(u, f) \in L^2 \times L^2$,

3) for any fixed $f \in L^2$, for each $u \in L^2$ there exists $F'_u(u,f) \in H_0^1$ such that

$$\frac{|F(u+v,f)-F(u,f)-(F'_u(u,f),v)|}{\|v\|} \to 0 \quad \text{if } \|v\| \to 0,\ v \in H_0^1 .$$

4) the mapping $F'_u\colon L^2 \times L^2 \to H_0^1$ is continuous and bounded.

Definition 2. A family $\{v_t\}_{t \in (0,T)} (T<\infty)$ of Borel probability measures on $L^2 \times L^2$ will be called *basic* if

1) $\mathrm{supp}(v_t) \subset H_0^1 \times L^2$ for each $t \in (0,T)$,

2) the function

$$t \to \int_{H_0^1 \times L^2} \{\|u\|_1^2 + \|f\|^2\} dv_t(u,f)$$

is well defined and integrable on $(0,T)$,

3) for some constant c_4 we have

$$\int_{H_0^1 \times L^2} \{\|u\|^2 + \|f\|^2\} dv_t(u,f) < c_4 \quad \text{for } t \in (0,T)$$

We fix any Borel probability measure v_0 on $L^2 \times L^2$ satisfying

$$\int_{L^2 \times L^2} \{\|u\|^2 + \|f\|^2\} dv_0(u,f) < \infty .$$

Definition 3. A basic family of measures $\{v_t\}_{t \in (0,T)}$ will be called a *statistical solution of the heat equation* with initial measure v_0 if and only if for every test functional F the following equality holds for each $s \in (0,T)$

$$\text{(1)} \quad \int_{H_0^1 \times L^2} F(u,f) dv_s(u,f) - \int_{L^2 \times L^2} F(u,f) dv_0(u,f) + $$
$$+ \int_0^s \int_{H_0^1 \times L^2} (u, F'_u(u,f))_1 dv_t(u,f) dt = \int_0^s \int_{H_0^1 \times L^2} (f, F'_u(u,f)) dv_t(u,f) dt .$$

3. Some properties of statistical solutions. The following theorem shows that the weak solution of the heat equation (see [3]) is also a statistical solution.

THEOREM 1. *The following conditions are equivalent*

(i) *the mapping* $u\colon (0,T) \to H_0^1$ *is the weak solution of the heat equation with initial value* u_0 *and right side term* f_0, *i.e. for every* $v \in C([0,T), H_0^1)$ *such that* $v'_t \in L^2((0,T), L^2)$, *for every* $s \in (0,T)$ *we have*

$$\text{(2)} \quad (u(s), v(s)) - (u_0, v(0)) + \int_0^s (u(t), v'_t(t)) dt + \int_0^s (u(t), v(t))_1 dt = \int_0^s (f_0, v(t)) dt ,$$

(ii) *the family of measures* $\{\delta_{u(t)} \times \delta_{f_0}\}$ *is a statistical solution of the heat equation with initial measure* $\delta_{u_0} \times \delta_{f_0}$.

The proof of this theorem as well as that of the next may be found in [1].

THEOREM 2. *There exists at most one statistical solution of the heat equation corresponding to the initial measure with bounded support in* $L^2 \times L^2$.