

## On the existence of an invariant measure for a quasi-linear partial differential equation

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**0. Introduction.** The problem of the existence of an invariant measure is important because of the connections between turbulence and the existence of an invariant measure. These connections were noticed by Prodi [8], Foias [4] and Hopf [5], who like Butler [2] dealt with the turbulence of trajectories of dynamical systems generated by partial differential equations. The method of construction of an invariant measure applied in the paper [3] comes from Avez [1] and Lasota and Pianigiani [7] Professor Lasota [6] has also dealt with the equation considered in paper [3] and in this paper.

In this paper I should like to expound paper [3], in which a theorem on existence of an invariant measure for a linear partial differential equation was proved. I have presented a generalization of this theorem for a quasi-linear equation. In the last section I have tried to prove a theorem on the stability of the solution of this equation.

### 1. Invariant measure for a dynamical system given by partial differential equations.

In this chapter  $X$  is understood as the space functions  $x: [0, 1] \rightarrow R$  satisfying the Lipschitz condition and such that  $x(0) = 0$ .  $X_+$  is understood as the space of non-negative valued functions of  $X$  and  $X_a$  as the space of all functions of  $X$  valued on  $[0, a]$  when  $a$  may be finite or not.

Remark. If  $a = \infty$  then  $X_a = X_+$

$X$ ,  $X_+$  and  $X_a$  will be considered with a topology generated by sets

$$U(x_0; v_0, \varepsilon) = \{v \in X \mid \exists c \in R: \forall x > x_0 |v(x) - v_0(x) - c| < \varepsilon\}.$$

Let us consider the differential equation

$$(1) \quad \frac{\partial u}{\partial t} = \lambda u - x \frac{\partial u}{\partial x}$$

$$(2) \quad t \geq 0, \quad 0 \leq x \leq 1$$

with boundary conditions

$$(3) \quad \begin{aligned} u(t; 0) &= 0, & t \geq 0 \\ u(0; x) &= v(x), & 0 \leq x \leq 1, \end{aligned}$$

where  $v \in X$  is a given function.

Let the semi-dynamical system

$$(4) \quad T_t: X \rightarrow X$$

be given by the formula

$$(5) \quad (T_t v)(x) = u(t; x).$$

**THEOREM 1.** *If  $\lambda > 1$ , then there exists a measure  $\mu$  on  $X$  satisfying the following conditions*

- (i)  $\mu$  is probabilistic,
- (ii)  $\mu$  is  $T_t$ -invariant,
- (iii)  $\mu(E) > 0$  for each open non-empty set  $E \subset X$ ,
- (iv)  $\mu$  is ergodic,
- (v)  $\mu(E_0) = 0$  where  $E_0$  is the set of all functions  $v$  such that  $\exists t > 0: |T_t v| = |v|$ .

This theorem is proved in the paper [3] but the point (v) is slightly different. This difference does not influence the proof.

Let us consider the same differential equation with the same boundary condition but let the semi-dynamical system

$$(6) \quad T_t^*: X_+ \rightarrow X_+$$

be given on  $X_+$ .

**THEOREM 2.** *If  $\lambda > 2$ , then there exists a measure  $\mu^*$  satisfying the following conditions*

- (i\*)  $\mu^*$  is probabilistic,
- (ii\*)  $\mu^*$  is  $T_t^*$ -invariant,
- (iii\*)  $\mu^*(E) > 0$  for each open non-empty set  $E \subset X_+$ ,
- (iv\*)  $\mu^*$  is ergodic,
- (v\*)  $\mu^*(E_0) = 0$  where  $E_0$  is the set of all periodic points.

Let us consider, finally, an equation

$$(7) \quad \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = F(u),$$

where  $F: U \rightarrow R$  ( $U$  is open and  $U \supset [0; a]$ ) satisfies the following conditions

- (a)  $F$  is continuous,
- (b)  $F(0) = F(a) = 0$ ,
- (c)  $F(x) > 0$  for every  $x \in (0; a)$ ,
- (d)  $F$  satisfies the Lipschitz condition,
- (e)  $F$  is of class  $C^2$  in the neighbourhood of 0.

In this situation the equation (7) generates a semi-dynamical system.

**THEOREM 3.** *If  $F'(0) = \lambda > 2$  then there exists a measure  $\nu$  satisfying the following conditions*

- (i)  $\nu$  is probabilistic,
- (ii)  $\nu$  is  $S_t$ -invariant,

- (iii)  $v(E) > 0$  for each open non-empty set  $E \subset X_a$ ,
- (iv)  $v$  is ergodic,
- (v)  $v(E_0) = 0$  where  $E_0$  is the set of all periodic points.

**2. Proof of theorem 2.**

Let us consider  $V_t: X \rightarrow X$  defined by the formula

$$(9) \quad (V_t u)(x) = e^{\frac{\lambda}{2}t} (xe^{-t}).$$

Since  $\frac{\lambda}{2} > 1$  it is obvious that there exists a measure  $\mu$  satisfying the thesis of theorem 1.

Let us define  $h: X \rightarrow X_+$  by the formula

$$(10) \quad h(u)(x) = [u(x)]^2$$

Now it will be shown that the diagram

$$(11) \quad \begin{array}{ccc} X & \xrightarrow{V_t} & X \\ \downarrow h & & \downarrow h \\ X_+ & \xrightarrow{T_t^*} & X_+ \end{array}$$

is commutative

$$\text{Let } v \in X, h(V_t v(x)) = [e^{\frac{\lambda}{2}t} v(xe^{-t})]^2 = e^{\lambda t} [v(xe^{-t})]^2 = e^{\lambda t} h(v)(xe^{-t}) = T_t^* h v(x).$$

This completes the claim.

The continuity of  $h$  results from the continuity of function  $y = x^2$ .

Define the measure  $\mu^*$  by the formula

$$(12) \quad \mu^*(E) = \mu(h^{-1}(E)).$$

It is obvious that the measure  $\mu^*$  satisfies the conditions (i\*), (ii\*), (iv\*). To prove the condition (iii\*) it is sufficient to prove that  $h(X)$  is dense in  $X_+$ . Let  $v$  then be an arbitrary function in  $X_+$ .

Define  $\bar{v}_n(x) = \max\left\{v(x); \frac{1}{n}\right\}$ . It is obvious that  $\bar{v}_n \xrightarrow{\rightarrow} v$ . Let us define  $v_n$  by the formula

$$(13) \quad v_n(x) = \begin{cases} n^2 \bar{v}_n \left(\frac{1}{n}\right) x^2 & \text{for } x \leq \frac{1}{n} \\ \bar{v}_n(x) & \text{for } x > \frac{1}{n} \end{cases}$$

Since  $v_n(x) = \bar{v}_n(x)$  for all  $x \in \left(\frac{1}{n}; 1\right]$ , it is obvious that  $v_n$  converges to  $v$  in the topology of  $X_+$ .