On the existence of an invariant measure for a quasi-linear partial differential equation

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0. Introduction. The problem of the existence of an invariant measure is important because of the connections between turbulence and the existence of an invariant measure. These connections were noticed by Prodi [8], Foiaş [4] and Hopf [5], who like Butler [2] dealt with the turbulence of trajectories of dynamical systems generated by partial differential equations. The method of construction of an invariant measure applied in the paper [3] comes from Avez [1] and Lasota and Pianigiani [7] Professor Lasota [6] has also dealt with the equation considered in paper [3] and in this paper.

In this paper I should like to expound paper [3], in which a theorem on existence of an invariant measure for a linear partial differential equation was proved. I have presented a generalization of this theorem for a quasi-linear equation. In the last section I have tried to prove a theorem on the stability of the solution of this equation.

1. Invariant measure for a dynamical system given by partial differential equations. In this chapter X is understood as the space functions x: $[0, 1] \to R$ satisfying the Lipschitz condition and such that $x(0) = 0$. X_+ is understood as the space of non-negative valued functions of X and X_a as the space of all functions of X valued on $[0, a)$ when a may be finite or not.

Remark. If $a = \infty$ then $X_a = X_+$
X, X_+ and X_a will be considered with a topology generated by sets

$$U(x_0; v_0, \varepsilon) = \{v \in X \mid \exists c \in R\colon \forall x > x_0 |v(x) - v_0(x) - c| < \varepsilon\}.$$

Let us consider the differential equation

$$\frac{\partial u}{\partial t} = \lambda u - x\frac{\partial u}{\partial x} \tag{1}$$

$$t \geqq 0, \quad 0 \leqq x \leqq 1 \tag{2}$$

with boundary conditions

$$u(t; 0) = 0, \quad t \geqq 0 \tag{3}$$

$$u(0; x) = v(x), \quad 0 \leqq x \leqq 1,$$

where $v \in X$ is a given function.

Let the semi-dynamical system

$$T_t\colon X \to X \tag{4}$$

be given by the formula

$$(T_t v)(x) = u(t; x)\,. \tag{5}$$

THEOREM 1. If $\lambda > 1$, *then there exists a measure* μ *on* X *satisfying the following conditions*

(i) μ *is probabilistic,*
(ii) μ *is* T_t*-invariant,*
(iii) $\mu(E) > 0$ *for each open non-empty set* $E \subset X$,
(iv) μ *is ergodic,*
(v) $\mu(E_0) = 0$ *where* E_0 *is the set of all functions* v *such that* $\exists t > 0\colon |T_t v| = |v|$.

This theorem is proved in the paper [3] but the point (v) is slightly different. This difference does not influence the proof.

Let us consider the same differential equation with the same boundary condition but let the semi-dynamical system

$$T_t^*\colon X_+ \to X_+ \tag{6}$$

be given on X_+.

THEOREM 2. *If* $\lambda > 2$, *then there exists a measure* μ^* *satisfying the following conditions*

(i*) μ^* *is probabilistic,*
(ii*) μ^* *is* T_t^**-invariant,*
(iii*) $\mu^*(E) > 0$ *for each open non-empty set* $E \subset X_+$,
(iv*) μ^* *is ergodic,*
(v*) $\mu^*(E_0) = 0$ *where* E_0 *is the set of all periodic points.*

Let us consider, finally, an equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = F(u)\,, \tag{7}$$

where $F\colon U \to R$ (U is open and $U \supset [0; a]$) satisfies the following conditions

(a) F is continuous,
(b) $F(0) = F(a) = 0$,
(c) $F(x) > 0$ for every $x \in (0; a)$,
(d) F satisfies the Lipschitz condition,
(e) F is of class C^2 in the neighbourhood of 0.

In this situation the equation (7) generates a semi-dynamical system.

THEOREM 3. *If* $F'(0) = \lambda > 2$ *then there exists a measure* ν *satisfying the following conditions*

(i) ν *is probabilistic,*
(ii) ν *is* S_t*-invariant,*

(iii) $\nu(E)>0$ *for each open non-empty set* $E\subset X_a$,
(iv) ν *is ergodic*,
(v) $\nu(E_0)=0$ *where* E_0 *is the set of all periodic points.*

2. Proof of theorem 2.

Let us consider $V_t\colon X\to X$ defined by the formula

$$(V_t u)(x) = e^{\frac{\lambda}{2}t}(xe^{-t})\,. \tag{9}$$

Since $\frac{\lambda}{2}>1$ it is obvious that there exists a measure μ satisfying the thesis of theorem 1.

Let us define $h\colon X\to X_+$ by the formula

$$h(u)(x) = [u(x)]^2 \tag{10}$$

Now it will be shown that the diagram

$$\begin{tikzcd} X \arrow[r, "V_t"] \arrow[d, "h"] & X \arrow[d, "h"] \\ X_+ \arrow[r, "T_t^*"] & X_+ \end{tikzcd} \tag{11}$$

is commutative

Let $v\in X$, $h\big(V_t v(x)\big) = [e^{\frac{\lambda}{2}}tv(xe^{-t})]^2 = e^{\lambda t}[v(xe^{-t})]^2 = e^{\lambda t}h(v)(xe^{-t}) = T_t^* hv(x)$.
This completes the claim.
The continuity of h results from the continuity of function $y=x^2$.
Define the measure μ^* by the formula

$$\mu^*(E) = \mu\big(h^{-1}(E)\big)\,. \tag{12}$$

It is obvious that the measure μ^* satisfies the conditions (i*), (ii*), (iv*). To prove the condition (iii*) it is sufficient to prove that $h(X)$ is dense in X_+. Let v then be an arbitrary function in X_+.

Define $\bar{v}_n(x) = \max\left\{v(x); \frac{1}{n}\right\}$. It is obvious that $\bar{v}_n \rightrightarrows v$. Let us define v_n by the formula

$$v_n(x) = \begin{cases} n^2\bar{v}_n\left(\frac{1}{n}\right)x^2 & \text{for } x\leq\frac{1}{n} \\ \bar{v}_n(x) & \text{for } x>\frac{1}{n} \end{cases}. \tag{13}$$

Since $v_n(x) = \bar{v}_n(x)$ for all $x\in\left(\frac{1}{n};1\right]$, it is obvious that v_n converges to v in the topology of X_+.

Since $v_n(x) \geqq \frac{1}{n}$ for $x \geqq \frac{1}{n}$, and $\sqrt{v_n(x)} = n\sqrt{v_n\left(\frac{1}{n}\right)}x$ for $x \leqq \frac{1}{n}$; so $\sqrt{v_n(x)}$ satisfies the Lipschitz condition on $[0; 1]$. This completes the proof of density $h(x)$ in X_+.

Let $T_t^* h(u) = h(u)$ for some $t > 0$. This means that $h(V_t u) = h(u)$, and from this it results that $|V_t u(x)| = |u(x)|$ which completes the proof.

3. Auxiliary lemmae.

LEMMA 1. *Let us consider the differential equation*

$$x' = \lambda x + \mu x^2 . \tag{14}$$

Let φ be a solution of this equation such that $\varphi(0) = x_0$. Let us consider the function

$$p(z) = \varphi\left(\frac{\ln z}{\lambda}\right). \tag{15}$$

Thus $\lim\limits_{z \to 0} \frac{p(z)}{z} = \frac{\lambda x_0}{\lambda + \mu x_0}$.

Proof. The solution of the equation (14) is given by the formula

$$\varphi(x) = \frac{\lambda x_0 e^{\lambda t}}{\lambda + \mu x_0 - \mu x_0 e^{\lambda t}}.$$

Hence,

$$p(z) = \frac{\lambda x_0 z}{\lambda + \mu x_0 - \mu x_0 z},$$

and in consequence

$$\lim_{z \to 0} \frac{p(z)}{z} = \frac{\lambda x_0}{\lambda + \mu x_0}.$$

LEMMA 2. *Let the function F satisfy the conditions* (a), (b), (c), (d), (e). *Let us consider the equation*

$$u' = F(u) . \tag{16}$$

Then there exists a solution of (16), *such that for p defined by the formula* (15) *we have*

$$0 < \lim_{z \to 0} \frac{p(z)}{z} \leqq \lim_{z \to 0} \frac{p(z)}{z} < \infty .$$

Proof. Let U be some neighbourhood of 0 and let $\mu_1 = \inf\{F''(x) | x \in U\}$ and $\mu_2 = \sup\{F''(x) | x \in U\}$. Then, for $x \in [0; a] \cap U$

$$\lambda x + \mu_1 x^2 \leqq F(x) \leqq \lambda x + \mu_2 x^2 \tag{17}$$

Let us consider $x_0 \in U$, such that $\lambda + \mu_1 x_0 > 0$, $\lambda + \mu_2 x_0 > 0$, $x_0 > 0$. Let φ be a solution of the equation (16) such that $\varphi(0) = x_0$. Let φ_i be a solution of equation

$$x' = \lambda x + \mu_i x^2, \quad (i = 1, 2) \tag{18}$$

such that $\varphi_i(0) = x_0$. From the monotonity of φ and φ_i and from the Czaplygin–Perron theorem [9] we have

$$\varphi_2(x) \leqq \varphi(x) \leqq \varphi_1(x) \quad \text{for } x<0$$

and in consequence

$$p_2(z) \leqq p(z) \leqq p_1(z) \quad \text{for } z<1$$

when $p_i(z) = \varphi_i\left(\frac{\ln z}{\lambda}\right)$, $\quad (i = 1, 2)$.

Since $z>0$ and $\frac{p_2(z)}{z} \leqq \frac{p(z)}{z} \leqq \frac{p_1(z)}{z}$ for $z<1$.

So the truth of lemma 2 results from lemma 1.

4. Proof of theorem 3.

Let us consider the dynamical system $U_t\colon (0; a) \to (0; a)$ generated by the equation (16). Let φ be a solution of (16) as in lemma 2. It may be proved that φ is a surjective function on $(0; a)$. Let $m_t(u) = e^{\lambda t}u$. It is obvious that $m_t\colon R_+ \to R_+$. Now we shall prove that the diagram

(19)
$$\begin{array}{ccc} R_+ & \xrightarrow{m_t} & R_+ \\ \downarrow p & & \downarrow p \\ (0; a) & \xrightarrow{U_t} & (0; a) \end{array}$$

is commutative. Indeed,

$$U_t p(z) = U_t \varphi\left(\frac{\ln z}{\lambda}\right) = \varphi\left(\frac{\ln z}{\lambda} + t\right) = \varphi\left(\frac{\ln z + \lambda t}{\lambda}\right) = \varphi\left(\frac{\ln z e^{\lambda t}}{\lambda}\right) = p(ze^{\lambda t}) = m_t z\,.$$

It may be proved that S is given by the formula

(20) $$(S_t v)(x) = U_t v(xe^{-t})\,.$$

Let $p^*\colon X_+ \to X_a$ be given by the formula $(p^*u)(x) = p(u(x))$. The correctness and surjectivity of p^* results from lemma 2 and the monotonity of p.
The commutativity of diagram

(20)
$$\begin{array}{ccc} X_+ & \xrightarrow{T_t^*} & X_+ \\ \downarrow p^* & & \downarrow p^* \\ X_a & \xrightarrow{S_t} & X_a \end{array}$$

results from the commutativity of diagram (19).

Now let μ^* be a measure from theorem 2. Let us define $\nu(E) = \mu^*(p^{*-1}(E))$. It is obvious that ν satisfies (i)–(v).

5. Some remarks.

Let us now consider the dynamical system from theorem 3.

Remark. Let φ_1 and φ_2 be two solutions of 16. Let $p_i(z) = \varphi_i\left(\frac{\ln z}{\lambda}\right)$, $(i = 1, 2)$. Then the existence of $\lim\limits_{z\to 0}\frac{p_1(z)}{z}$ is the necessary and sufficient condition for the existence of $\lim\limits_{z\to 0}\frac{p_2(z)}{z}$.

Proof. Since φ_1 and φ_2 are solutions of (16), and (16) is an autonomic equation, then there exists t_0, such that $\varphi_2(t) = \varphi_1(t-t_0)$. Hence

$$\frac{p_2(z)}{z} = \frac{1}{z}\varphi_2\left(\frac{\ln z}{\lambda}\right) = \frac{1}{z}\varphi_1\left(\frac{\ln z}{\lambda}-t_0\right) = \frac{1}{z}\varphi_1\left(\frac{\ln ze^{-\lambda t_0}}{\lambda}\right) = \frac{p_2(ze^{-\lambda t_0})}{ze^{-\lambda t_0}}e^{\lambda t_0}.$$

Since $e^{\lambda t_0}$ is constant, the remark has been proved.

Let us now consider the equation (7) with the boundary conditions (3). Let $\lim\limits_{z\to 0}\frac{p(z)}{z} = p_0$ (of p in the lemmae) i.e. it exists for every p.

THEOREM 4.

a) *If* $\lim\limits_{x\to 0}\frac{v(x)}{x^\lambda} = 0$, *then* $\lim\limits_{t\to 0} S_t v(x) = 0$,

b) *If* $\lim\limits_{x\to 0}\frac{v(x)}{x^\lambda} = g>0$, *then* $\lim\limits_{t\to\infty} S_t v(x) = \varphi(\ln x + A)$ *for some solution* φ *of* (16) *and some* A,

c) *If* $\lim\limits_{x\to 0}\frac{v(x)}{x^\lambda} = \infty$, *then* $\lim\limits_{t\to\infty}(S_t v)(x) = a$.

Proof. c) Let $B>0$. If $\lim\limits_{x\to 0}\frac{v(x)}{x^\lambda} = \infty$, then there is δ such as for all $x<\delta v(x)>Bx^\lambda$. Hence $v(xe^{-t})>Bx^\lambda e^{-\lambda t}$ for sufficiently large t, not depending on x. In this case

$$S_t v(x) = U_t v(xe^{-t}) = p(e^{\lambda t}p^{-1}(v(xe^{-t})))\geqq p(e^{\lambda t}p^{-1}(Bx^\lambda e^{-\lambda t}))\geqq p\left(\frac{Bx}{p_0+\varepsilon}\right)$$

if t is sufficiently large.

a) Let $\varepsilon>0$. For sufficiently large t $v(xe^{-t})<\varepsilon x^\lambda e^{-\lambda t}$. Then $S_t v(x)\leqq p\left(\frac{\varepsilon x^\lambda}{p_0-\varepsilon}\right)$ for sufficiently large t, and hence $S_t v\to 0$.

b) Let $\varepsilon>0$. For sufficiently large t $p\left(\frac{g-\varepsilon}{p_0+\varepsilon}x^\lambda\right)\leqq(S_t v)(x)\leqq p\left(\frac{g+\varepsilon}{p_0-\varepsilon}x^\lambda\right)$. Hence

$$(S_t v)(x)\to p\left(\frac{g}{p_0}x^\lambda\right) = \varphi\left(\frac{1}{\lambda}\ln\frac{g}{p_0}+\ln x\right).$$

This completes the proof.

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