ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DCXXIII PRACE MATEMATYCZNE, ZESZYT 23 1982

Differentiability of solutions of elliptic problems with respect to coefficients

by Anna KRAKOWIAK

1. In the present note we are concerned with the question of differentiability of weak solutions of elliptic Dirichlet problems with respect to coefficients in the sense of some norms.

Our method of the proof is similar to that of [2]. Schechter in [2] gives a theorem on differentiability of strong solutions of elliptic problems with respect to parameters.

2. Let $\Omega \subset R^n$ be a bound domain with C^{2m} boundary $\partial\Omega$ and let $H^m(\Omega)$ $(m \geqslant 1)$ denote the space of those functions $u(x) \in L^2(\Omega)$ which satisfy $D^\alpha u \in L^2$ for all $|\alpha| \leqslant m$. Here $D^\alpha = \partial^{\alpha_1+,,,+\alpha_n}. (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and the derivatives are taken in the sense of the theory of distributions).

The norm and scalar product are defined by the formulae

$$\|u(x)\|_m = \Big(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha u(x)|^2 dx \Big)^{\frac{1}{2}},$$

$$\big(u(x), v(x)\big) = \sum_{|\alpha| \leqslant m} \int_\Omega D^\alpha u(x) \cdot \overline{D^\alpha v(x)}\, dx .$$

We shall denote by $\mathring{H}^m$ the closure of C_0^∞ in H^m, where C_0^∞ is the set of all C^∞ functions with compact support in Ω. Moreover we shall use the following notations:

$(\, ,)$ is scalar product in L^2,

$\mathring{H}^{m*}$ is adjoint to $\mathring{H}_m$,

$\| \; \|_{m*}$ is norm in $\mathring{H}^{m*}$,

$\langle f, \varphi \rangle$ is value of the functional $f \in \mathring{H}^{m*}$ for $\varphi \in \mathring{H}^m$.

3. Consider on elliptic equation

$$A_v[u(x)] = \sum_{\substack{0 \leqslant \delta \leqslant m \\ 1 \leqslant \varrho \leqslant m}} (-1)^{|\varrho|} D^\varrho\big(a^{\varrho\delta}(x)\, D^\delta u(x)\big) + v(x) u(x) = f(x) \tag{1}$$

Denote by

$$B_v(u, \varphi) = \sum_{\substack{0 \leqslant \delta \leqslant m \\ 1 \leqslant \varrho \leqslant m}} (D^\varrho \varphi, a^{\delta\varrho} D^\delta u) + \big(\varphi, v(x) u\big)$$

We assume that

(i) The operator A_v is strongly elliptic in Ω (i.e. $\operatorname{Re}\big(\sum_{|\varrho|=|\delta|=m} \xi^{\varrho} a^{\varrho\delta}(x)\xi^{\delta}\big) \geq C_0|\xi|^{2m}$, $C_0>0$, for every $\xi, x \in \Omega$).

(ii) The coefficients $a^{\varrho\delta}$ are bounded in Ω and $v(x)$ is continuous in $\overline{\Omega}$.

(iii) The coefficients $a^{\varrho\delta}$ for $|\varrho| = |\delta| = m$ have got the module of continuity $c_2(t)$ in Ω (i.e. $|a^{\varrho\delta}(x)-a^{\varrho\delta}(y)| \leqslant c_2(|x-y|)$ for $x \in \Omega$, $y \in \Omega$; $c_2(t)\searrow 0$ if $t\searrow 0$).

(iv) The bilinear form satisfies the condition $\operatorname{Re} B_v(\varphi, \varphi) \geqslant c|\varphi|_m^2$ for every $\varphi \in \mathring{H}^m$ and v from a neighborhood V of v_0.

(In the set of all continuous mappings C we define the usual norm, $\|v\|_C = \max\{|v(x)|: x \in \overline{\Omega}\}$).

Let $f \in \mathring{H}^{m*}$. We shall say that $u_v \in \mathring{H}^m$ is a weak solution of Dirichlet problem of the equation (1) if the equality

$$B_v(u_v, \varphi) = \langle f, \varphi\rangle \tag{2}$$

is satisfied for every $\varphi \in \mathring{H}^m$.

Under the assumptions (i)–(iv) there exists the exactly one weak solution of Dirichlet problem of equation (1) for every $v \in V$ ([1], theorem 6, p. 363).

We have the following

THEOREM. The weak solution u_v is infinitely differentiable with respect to v for $v = v_0$.

Proof. If u and v are fixed, then the bilinear form $B_v(u, \cdot)$ is a bounded linear functional on $\mathring{H}^m$. By the well-known Riesz theorem on the form of continuous linear functionals there exists some element $z \in \mathring{H}^m$, such that

$$B_v(u, \varphi) = (\varphi, z)_m, \qquad \|B_v(u, \cdot)\|_{m*} = \|z\|_m.$$

Setting

$$z = A_v u$$

from [1], p. 362 it follows that A_v is the linear homeomorphic map $\mathring{H}^m$ onto $\mathring{H}^m$. Hence there exist constants $k_1, k_2>0$ such that the inequality

$$k_2\|u\|_m \geqslant \|B(u, \cdot)\|_{m*} = \|A_v u\|_m \geqslant k_1\|u\|_m \tag{3}$$

is satisfied.

Denoting

$$L_v\colon u \in \mathring{H}^m \to B_v(u, \cdot) \in \mathring{H}^{m*}$$

we conclude by (3), that l_v is the linear homeomorphic map of the space $\mathring{H}^m$ onto $\mathring{H}^{m*}$. The equation (2) can be written as follows

$$L_v u_v = f. \tag{4}$$

Note that

$$L_{v0}^{-1}L_v = L_{v0}^{-1}(L_v - L_{v0} + L_{v0}) = L_{v0}^{-1}(L_v - L_{v0}) + I \tag{5}$$

Since

$$\begin{aligned}\|L_{v0}^{-1}(L_v - L_{v0})u\|_m &\leqslant \|L_{v0}^{-1}\|\,\|(L_v - L_{v0})u\|_{m*}\\ &= \|L_{v0}^{-1}\|\,\|(\cdot, v(x)-v_0(x)u)\|_{m*} \leqslant \|L_{v0}^{-1}\|\,\|(v(x)-v_0(x))u\|_{L^2}\\ &\leqslant \|L_{v0}^{-1}\|\,(\max|v(x)-v_0(x)|)\|u\|_m\end{aligned}$$

we have by (5) that there exists a constant C such that

$$\|L_{v0}^{-1}L_v\| \leqslant C \quad \text{for every } v \in V. \tag{6}$$

From (4) we have

$$L_{v0}^{-1}L_v u_v = L_{v0}^{-1}f$$

and consequently

$$u_v = (L_{v0}^{-1}L_v)^{-1}(L_{v0}^{-1}f).$$

By (6), there exists a constant C_1 such that

$$\|u_v\|_m \leqslant \|(L_{v0}^{-1}L_v)^{-1}\| \, \|L_{v0}^{-1}f\| \leqslant C_1; \quad \text{for every } v \in V. \tag{7}$$

Now, let us consider

$$\begin{aligned} u_v - u_{v0} &= (L_{v0}^{-1}L_{v0})(u_v - u_{v0}) = L_{v0}^{-1}L_{v0}u_v - L_{v0}^{-1}L_{v0}u_{v0} \\ &= L_{v0}^{-1}L_{v0}u_v - L_{v0}^{-1}f = L_{v0}^{-1}L_{v0}u_v - L_{v0}^{-1}L_v u_v \\ &= L_{v0}^{-1}(L_{v0} - L_v)u_v. \end{aligned} \tag{8}$$

We have also

$$\|u_v - u_{v0}\|_m \leqslant \|L_{v0}^{-1}\| \big(\max_{\bar\Omega} |v(x) - v_0(x)|\big) \|u_v\|_m.$$

Hence (7) implies

$$\|u_v - u_{v0}\|_m \to 0 \quad \text{as} \quad \|v(x) - v_0(x)\|_C \to 0. \tag{9}$$

Denote by $(u_v)'_{v=v_0}$ the Fréchet derivative of u_v for $v = v_0$ with respect to $v(x)$. We shall show that for any continuous function $h(x)$

$$(u_v)'_{v=v_0}(h(x)) = L_{v0}^{-1}(\cdot, -h(x)u_{v_0}(x)).$$

In fact, by (8)

$$\begin{aligned} &\frac{\|u_v - u_{v_0} - L_{v0}^{-1}(\cdot, (v_0(x) - v(x))u_{v_0}(x))\|_m}{\|v(x) - v_0(x)\|_C} \\ &= \frac{\|L_{v0}^{-1}(\cdot, (v_0(x) - v(x))u_v) - L_{v0}^{-1}(\cdot, (v_0(x) - v(x))u_{v0}\|_m}{\|v(x) - v_0(x)\|_C} \\ &= \frac{\|L_{v0}^{-1}(\cdot, (v_0(x) - v(x))(u_v - u_{v0}))\|_m}{\|v(x) - v_0(x)\|_C} \leqslant \frac{\|L_{v0}^{-1}\| \, \|v - v_0\|_C \, \|u_v - u_{v0}\|_m}{\|v(x) - v_0(x)\|_C} \end{aligned}$$

which tends to 0, as $\|v - v_0\|_C \to 0$, because of (9). To prove that u'_v is differentiable with respect v we differentiate equation (2) with respect to v and have that u'_v satisfies the equation

$$\sum_{\substack{0 \leqslant |\delta| \leqslant m \\ 1 \leqslant |\varrho| \leqslant m}} \big(D^\varrho \varphi, a^{\varrho\delta} D^\varrho u'_v(h(x))\big) + (\varphi, h(x)u_v) + \big(\varphi, v(x)u'_v(h(x)\big) = 0.$$

Thus

$$\sum_{\substack{0 \leqslant |\delta| \leqslant m \\ 1 \leqslant |\varrho| \leqslant m}} \big(D^\varrho \varphi, a^{\varrho\delta} D^\delta u'_v(h(x))\big) + \big(\varphi, v(x)u'_v(h(x)\big) = -(\varphi, h(x)u_v). \tag{10}$$

The equation (10) can be written as

$$L_v u_v'(h(x)) = (\cdot, -h(x)u_v).$$

Hence we obtain

$$L_{v0}^{-1} L_v u_v'(h(x)) = L_{v0}^{-1}(\cdot, -h(x)u_v). \tag{11}$$

From (11) it follows that

$$u_v'(h(x)) = (L_{v0}^{-1} L_v)^{-1} L_{v0}^{-1}(\cdot, -h(x)u_v)$$

and

$$\|u_v'(h(x))\|_m \leqslant \|(L_{v0}^{-1} L_v)^{-1}\| \; \|L_{v0}^{-1}\| \; \|h(x)\|_C \|u_v\|_m. \tag{12}$$

By (6), (7), (12) we have that there exists a constant C_2 such that

$$\|u_v'\| \leqslant \|(L_{v0}^{-1} L_v)^{-1}\| \; \|L_{v0}^{-1}\| \; \|u_v\|_m < C_2. \tag{13}$$

Notice that

$$\begin{aligned} u_v'(h(x)) - v_{v0}'(h(x)) &= L_{v0}^{-1} L_{v0}\big(u_v'(h(x)) - u_{v0}'(h(x))\big) \\ &= L_{v0}^{-1}\big(L_{v0} u_v'(h(x)) - L_{v0} u_{v0}'(h(x)) + L_v u_v'(h(x)) - L_v u_v'(h(x))\big) \\ &= L_{v0}^{-1}\big((L_{v0} - L_v) u_v'(h(x)) + (\cdot, (u_{v0} - u_v) h(x))\big) \\ &= L_{v0}^{-1}\big((\cdot, (v_0(x) - v(x)) u_v'(h(x))) + (\cdot, (u_{v0} - u_v) h(x))\big). \end{aligned} \tag{14}$$

From here we have

$$\|u_v'(h(x)) - u_{v0}'(h(x))\|_m \leqslant \|L_{v0}^{-1}\| \big(\|v_0 - v\|_C \|u_v'\| + \|u_{v0} - u_v\|_m\big) \|h(x)\|_C.$$

This implies

$$\|u_v' - u_{v0}'\| \leqslant \|L_{v0}^{-1}\| \big(\|v_0(x) - v(x)\|_C \|u_v'\| + \|u_{v0} - u_v\|_m\big)$$

(9), (13) shows that

$$\|u_v' - u_{v0}^{-}\| \to 0 \quad \text{as } \|v(x) - v_0(x)\|_C \to 0. \tag{15}$$

Now we shall show that for any continuous functions $h(x)$ and $g(x)$ the second Fréchet derivative satisfies

$$(u_v'')_{v=v0}(g(x))(h(x)) = L_{v0}^{-1}\big(\cdot, -g(x) u_{v0}(h(x)) + h(x) u_{x0}'(g(x))\big). \tag{16}$$

In fact, by (14)

$$\begin{aligned} &\|u_v'(h(x)) - u_{v0}'(h(x)) - L_{v0}^{-1}(\cdot, (v_0(x) - v(x)) u_{v0}'(h(x)) + h(x) u_{v0}'(v(x) - v_0(x)))\|_m \\ &= \|L_{v0}^{-1}\big(\cdot, \big((v_0(x) - v(x)) u_v'(h(x)) - (v_0(x) - v(x)) u_{v0}'(h(x))\big) + \\ &\quad + \big(\cdot, \big((u_{v0} - u_v) - u_{v0}'(v(x) - v_0(x)) h(x)\big)\big)\big)\|_m \\ &\leqslant \|L_{v0}^{-1}\| \big(\|v_0 - v\|_C \|u_v' - u_{v0}'\| + \|u_{v0}' - u_v - u_{v0}'(v(x) - v_0(x))\|_m\big) \| h(x)_C. \end{aligned}$$

Hence by (15) we obtain (16).

In a way very similar to the method used in the proof of the existence of the second derivative we may show that u_v is infinitely differentiable with respect to v for $v = v_0$.

Remark. A similar result may be obtained for the coefficients $a^{\varrho\delta}$ $0 \leqslant \delta \leqslant m$, $1 \leqslant \varrho \leqslant m$. In those cases the proof is analogous to the one presented above.

References

[1] A. Фридман, *Уравнения с частными производными параболического типа*, Москва 1968.
[2] M. Schechter, *Differentiability of solutions of elliptic problems with respect to parameters*, Boll. Un. Mat. Ital., (5) 13-A (1976), 601-608.

INSTITUTE OF MATHEMATICS
JAGELLONIAN UNIVERSITY
KRAKÓW (POLAND)