

A generalized solution of Oseen's equations

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The present paper is devoted to the existence and uniqueness of the generalized solution of Oseen's equations. In the first part we give the basic notations and definitions of the special spaces, of the operator A_e and of the functional b . We shall prove their properties. The generalized solution of Oseen's equations is defined in the second part. In this part we prove the existence and uniqueness theorems.

1. On some functions space

1.1. Let $\Omega \subset R^3$ be a bounded domain. We assume that the boundary $\partial\Omega$ of Ω is of class C^2 .

Let L^p ($1 \leq p < +\infty$) denote the space of measurable (with respect to the Lebesgue measure) vector-valued functions $u = (u_1, u_2, u_3)$ defined on Ω such that

$$|u|_p := \left\{ \int_{\Omega} \left[\sum_{i=1}^3 (u_i(x))^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} < +\infty.$$

For $u, v \in L^2$ we shall use the following notation

$$(u, v)_2 = \int \sum_{i=1}^3 u_i(x)v_i(x) dx.$$

$(,)_2$ is a scalar product and obviously $|u|_2^2 = (u, u)_2$.

Let H^k ($k \geq 1$) be a Sobolev space (see [5] p. 13) i.e. the space of those $u \in L^2$ for which $D^\alpha u = (D^\alpha u_1, D^\alpha u_2, D^\alpha u_3) \in L^2$ for all $|\alpha| \leq k$ ($D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and derivatives are taken in the sense of the theory of distributions).

In H^k we take the usual norm

$$\|u\|_k = \left(\sum_{|\alpha| \leq k} |D^\alpha u|_2^2 \right)^{\frac{1}{2}}.$$

H^k ($k > 1$) is the Hilbert space with the scalar product

$$((u, v))_k = \int_{\Omega} \sum_{i=1}^3 \sum_{|\alpha| \leq k} D^\alpha u_i(x) D^\alpha v_i(x) dx$$

(see [5] p. 15).

Let \mathcal{M} be the space of vector-valued functions $u = (u_1, u_2, u_3)$ defined on Ω such that $u_j \in C_0^\infty(\Omega)$ for $j = 1, 2, 3$ (i.e. $u_j: \Omega \rightarrow R$ is a C^∞ function with compact support

in Ω), $\operatorname{div} u = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = 0$ in Ω . We shall denote by N the closure of \mathcal{M} in L^2 and by N^1

the closure of \mathcal{M} in H^1 . For $k > 1$ we shall put

$$N^k = N^1 \cap H^k.$$

The sequence of spaces $\dots \subset N^2 \subset N^1 \subset N$ verifies the following conditions (see [6] p. 79) N^k is dense in N^{k-1} for $k > 1$, N^1 is dense in N .

The scalar product in N^1 can be also defined by

$$(1) \quad ((u, v)) = \int_{\Omega} \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dx \quad (\text{see [7] p. 62}).$$

We shall use the following notation

$$(2) \quad \|u\| = ((u, u)) \quad \text{for } u \in N^1.$$

According to Poincaré inequality (see [7] p. 62) we deduce that there exists a constant c_1 not depending on u , such that

$$(3) \quad \|u\|_1 \leq c_1 \|u\| \quad \text{for all } u \in N^1.$$

From the above inequality and definition of $\| \cdot \|_1$ it follows that the norms $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent.

Thanks to $\|u\|_1 = |u|_2 + \|u\|$ we have

$$(4) \quad |u|_2 \leq c_1 \|u\| \quad \text{for all } u \in N^1.$$

In sequel we shall always consider the space N^1 with the scalar product $((,))$ N^1 with this product is the Hilbert space.

Now, we introduce the conjugate space to N^1 , where the duality extends that given by $(u, v)_2$ (with $v \in N^1, u \in N$). Let u be any fixed element of N and let us consider a real, linear functional

$$(u, \cdot)_2: N^1 \ni v \rightarrow (u, v)_2 \in R.$$

From the inequality (4) we obtain

$$(4') \quad |(u, v)_2| \leq c_1 \|v\| |u|_2.$$

It means that this functional is continuous. According to Riesz theorem (see [9] p. 105) there exists the unique element $Iu \in N^1$ such that

$$(5) \quad (u, v)_2 = ((Iu, v)) \quad \text{for all } v \in N^1.$$

In such a way we define the linear, continuous operator $I: N \rightarrow N^1$. For $u, v \in N$ we put

$$(6) \quad (u, v)_{-1} := (Iu, v)_2 = ((Iu, Iv)).$$

It is a scalar product (see [2] p. 46). We shall denote by N^{-1} the complement to the Hilbert space of the space N with the scalar product $(\cdot, \cdot)_{-1}$.

For all $u \in N^1$ we have

$$|u|_{-1} \leq c_1 |u|_2 \leq c_1^2 \|u\| \quad (\text{see [2] p. 47}).$$

Let I_e denote the extension (by continuity) of I to the operator from N^{-1} into N . I_e is isometrical operator (see [2] p. 47).

For $\alpha \in N^{-1}$ and $u \in N^1$ we shall put

$$(7) \quad (\alpha, u)_2 = \lim_{n \rightarrow \infty} (u_n, u)$$

where $u_n \in N$ and $|u_n - \alpha|_{-1} \rightarrow 0$ for $n \rightarrow \infty$.

In case $\alpha \in N^{-1}$ and $u \in N^1$ the following inequality is satisfied

$$|(\alpha, u)_2| \leq c_1 |\alpha|_{-1} \|u\|.$$

From (6) we deduce that

$$\begin{aligned} (\alpha, \beta)_{-1} &= (\alpha, I_e \beta)_2 = ((I_e \alpha, I_e \beta)) \quad \text{for } \alpha, \beta \in N^{-1} \\ (\alpha, u)_2 &= ((I_e \alpha, u)) \quad \text{for } \alpha \in N^{-1}, u \in N^1. \end{aligned}$$

The space N^{-1} is identified with the space of all real, linear and continuous functionals defined on N^1 .

For all $\alpha \in N^{-1}$ we have

$$(8) \quad |\alpha|_{-1} = \sup_{u \in N^1} \frac{(\alpha, u)_2}{\|u\|}.$$

1.2. Let Δ denote the Laplace operator defined on the vector-valued functions $u = (u_1, u_2, u_3)$ i.e.

$$\begin{aligned} \Delta u &= (\Delta u_1, \Delta u_2, \Delta u_3) \\ \Delta u_i &= \sum_{j=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad i = 1, 2, 3. \end{aligned}$$

We consider the operator $D = -\Delta|_{\mathcal{M}}$ in the Hilbert space N . The domain $\mathcal{D}(D) = \mathcal{M}$ is dense in N . For all $u, v \in \mathcal{M}$ we have

$$\begin{aligned} (Du, v)_2 &= - \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} v_i dx = \int_{\Omega} \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx = ((u, v)) \\ &= ((v, u)) = (Dv, u)_2 = (u, Dv)_2 \end{aligned}$$

and

$$(Du, u)_2 = ((u, u)) = \|u\|^2 \geq \frac{1}{c_1^2} \|u\|_1^2 \geq \frac{1}{c_1^2} |u|_2^2 \quad (\text{from (3)}).$$

