

A generalized solution of Oseen's equations

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The present paper is devoted to the existence and uniqueness of the generalized solution of Oseen's equations. In the first part we give the basic notations and definitions of the special spaces, of the operator A_e and of the functional b . We shall prove their properties. The generalized solution of Oseen's equations is defined in the second part. In this part we prove the existence and uniqueness theorems.

1. On some functions space

1.1. Let $\Omega \subset R^3$ be a bounded domain. We assume that the boundary $\partial\Omega$ of Ω is of class C^2 .

Let L^p ($1 \leq p < +\infty$) denote the space of measurable (with respect to the Lebesgue measure) vector-valued functions $u = (u_1, u_2, u_3)$ defined on Ω such that

$$|u|_p := \left\{ \int_{\Omega} \left[\sum_{i=1}^3 (u_i(x))^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} < +\infty.$$

For $u, v \in L^2$ we shall use the following notation

$$(u, v)_2 = \int \sum_{i=1}^3 u_i(x)v_i(x) dx.$$

$(,)_2$ is a scalar product and obviously $|u|_2^2 = (u, u)_2$.

Let H^k ($k \geq 1$) be a Sobolev space (see [5] p. 13) i.e. the space of those $u \in L^2$ for which $D^\alpha u = (D^\alpha u_1, D^\alpha u_2, D^\alpha u_3) \in L^2$ for all $|\alpha| \leq k$ ($D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and derivatives are taken in the sense of the theory of distributions).

In H^k we take the usual norm

$$\|u\|_k = \left(\sum_{|\alpha| \leq k} |D^\alpha u|_2^2 \right)^{\frac{1}{2}}.$$

H^k ($k > 1$) is the Hilbert space with the scalar product

$$((u, v))_k = \int_{\Omega} \sum_{i=1}^3 \sum_{|\alpha| \leq k} D^\alpha u_i(x) D^\alpha v_i(x) dx$$

(see [5] p. 15).

Let \mathcal{M} be the space of vector-valued functions $u = (u_1, u_2, u_3)$ defined on Ω such that $u_j \in C_0^\infty(\Omega)$ for $j = 1, 2, 3$ (i.e. $u_j: \Omega \rightarrow \mathbb{R}$ is a C^∞ function with compact support

in Ω), $\operatorname{div} u = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = 0$ in Ω . We shall denote by N the closure of \mathcal{M} in L^2 and by N^1

the closure of \mathcal{M} in H^1 . For $k > 1$ we shall put

$$N^k = N^1 \cap H^k.$$

The sequence of spaces $\dots \subset N^2 \subset N^1 \subset N$ verifies the following conditions (see [6] p. 79) N^k is dense in N^{k-1} for $k > 1$, N^1 is dense in N .

The scalar product in N^1 can be also defined by

$$(1) \quad ((u, v)) = \int_{\Omega} \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx \quad (\text{see [7] p. 62}).$$

We shall use the following notation

$$(2) \quad \|u\| = ((u, u)) \quad \text{for } u \in N^1.$$

According to Poincaré inequality (see [7] p. 62) we deduce that there exists a constant c_1 not depending on u , such that

$$(3) \quad \|u\|_1 \leq c_1 \|u\| \quad \text{for all } u \in N^1.$$

From the above inequality and definition of $\| \cdot \|_1$ it follows that the norms $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent.

Thanks to $\|u\|_1 = |u|_2 + \|u\|$ we have

$$(4) \quad |u|_2 \leq c_1 \|u\| \quad \text{for all } u \in N^1.$$

In sequel we shall always consider the space N^1 with the scalar product $((,))$ N^1 with this product is the Hilbert space.

Now, we introduce the conjugate space to N^1 , where the duality extends that given by $(u, v)_2$ (with $v \in N^1, u \in N$). Let u be any fixed element of N and let us consider a real, linear functional

$$(u, \cdot)_2: N^1 \ni v \rightarrow (u, v)_2 \in \mathbb{R}.$$

From the inequality (4) we obtain

$$(4') \quad |(u, v)_2| \leq c_1 \|v\| |u|_2.$$

It means that this functional is continuous. According to Riesz theorem (see [9] p. 105) there exists the unique element $Iu \in N^1$ such that

$$(5) \quad (u, v)_2 = ((Iu, v)) \quad \text{for all } v \in N^1.$$

In such a way we define the linear, continuous operator $I: N \rightarrow N^1$. For $u, v \in N$ we put

$$(6) \quad (u, v)_{-1} := (Iu, v)_2 = ((Iu, Iv)).$$

It is a scalar product (see [2] p. 46). We shall denote by N^{-1} the complement to the Hilbert space of the space N with the scalar product $(\cdot, \cdot)_{-1}$.

For all $u \in N^1$ we have

$$|u|_{-1} \leq c_1 |u|_2 \leq c_1^2 \|u\| \quad (\text{see [2] p. 47}).$$

Let I_e denote the extension (by continuity) of I to the operator from N^{-1} into N . I_e is isometrical operator (see [2] p. 47).

For $\alpha \in N^{-1}$ and $u \in N^1$ we shall put

$$(7) \quad (\alpha, u)_2 = \lim_{n \rightarrow \infty} (u_n, u)$$

where $u_n \in N$ and $|u_n - \alpha|_{-1} \rightarrow 0$ for $n \rightarrow \infty$.

In case $\alpha \in N^{-1}$ and $u \in N^1$ the following inequality is satisfied

$$|(\alpha, u)_2| \leq c_1 |\alpha|_{-1} \|u\|.$$

From (6) we deduce that

$$\begin{aligned} (\alpha, \beta)_{-1} &= (\alpha, I_e \beta)_2 = ((I_e \alpha, I_e \beta)) \quad \text{for } \alpha, \beta \in N^{-1} \\ (\alpha, u)_2 &= ((I_e \alpha, u)) \quad \text{for } \alpha \in N^{-1}, u \in N^1. \end{aligned}$$

The space N^{-1} is identified with the space of all real, linear and continuous functionals defined on N^1 .

For all $\alpha \in N^{-1}$ we have

$$(8) \quad |\alpha|_{-1} = \sup_{u \in N^1} \frac{(\alpha, u)_2}{\|u\|}.$$

1.2. Let Δ denote the Laplace operator defined on the vector-valued functions $u = (u_1, u_2, u_3)$ i.e.

$$\begin{aligned} \Delta u &= (\Delta u_1, \Delta u_2, \Delta u_3) \\ \Delta u_i &= \sum_{j=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad i = 1, 2, 3. \end{aligned}$$

We consider the operator $D = -\Delta|_{\mathcal{M}}$ in the Hilbert space N . The domain $\mathcal{D}(D) = \mathcal{M}$ is dense in N . For all $u, v \in \mathcal{M}$ we have

$$\begin{aligned} (Du, v)_2 &= - \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} v_i dx = \int_{\Omega} \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx = ((u, v)) \\ &= ((v, u)) = (Dv, u)_2 = (u, Dv)_2 \end{aligned}$$

and

$$(Du, u)_2 = ((u, u)) = \|u\|^2 \geq \frac{1}{c_1^2} \|u\|_1^2 \geq \frac{1}{c_1^2} |u|_2^2 \quad (\text{from (3)}).$$

According to Friedrichs' theorem there exists the selfadjoint extension A (in N) of D (see [8]).

We have also $\mathcal{D}(A) = N^2$ (the domain), $\mathcal{R}(A) = N$ (the range) and

$$(Au, u)_2 \geq \frac{1}{c_1^2} |u|_2^2 \quad \text{for all } u \in N^2.$$

The space N^1 can be considered as the domain of selfadjoint operator A in N such that $A \geq 0$ and

$$((u, v)) = (Au, Av)_2 \quad \text{for all } u, v \in N^1$$

(see [5] p. 22).

For $u \in \mathcal{D}(A)$ and $v \in N^1$ we have

$$(Au, v)_2 = ((u, v)) = (Au, Av)_2 = (A \circ Au, v)_2.$$

Thus $A = A^\sharp$ and obviously $\mathcal{D}(A^\sharp) = N^1$.

Let A_e denotes the linear continuous operator from N^1 into N^{-1} extending A

$$(9) \quad (A_e u, v)_2 = ((u, v)) \quad \text{for all } u, v \in N^1.$$

1.3. Applying the theorem about inverse of linear and symmetrical operator the range and the domain of which are dense in the Hilbert space (see [9] p. 168) we deduce that there exists the symmetrical operator A^{-1} .

LEMMA 1. *The operator A^{-1} is compact in N .*

Proof. By Relich's theorem (see [9] p. 207) A^{-1} is compact in N if only if the set $\{u \in N; (Au, u)_2 \leq 1\}$ is relatively compact. We have

$$\{u \in N; (Au, u)_2 \leq 1\} = \{u \in N^2; \|u\| \leq 1\}$$

because $(Au, u)_2 = ((u, u))$ for all $u \in N^2$.

The bounded set in the space N^1 is compact in N because Ω is a bounded domain (see [7] p. 64).

Thus, A^{-1} is compact in N and it finishes the proof.

From above lemma and spectral theory (see [9] p. 216) it follows that there exists the orthonormal basis $\{\omega_m\}$ of N such that

$$A^{-1} \omega_n = \tilde{\lambda}_n \omega_n, \quad \text{where } \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_3 \geq \dots \geq \tilde{\lambda}_n \geq \dots \rightarrow 0.$$

Putting $\lambda_n = \frac{1}{\tilde{\lambda}_n}$ we obtain

$$A \omega_n = \lambda_n \omega_n, \quad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots \rightarrow +\infty.$$

We denote by P_m the orthogonal projection of N onto the subspace spanned by $\omega_1, \dots, \omega_m$ i.e.

$$P_m u = \sum_{k=1}^m (u, \omega_k) \omega_k \quad \text{for all } u \in N.$$

These projections P_m ($m = 1, 2, 3, \dots$) will play an essential role in the sequel.

LEMMA 2. Let $a \in N^1$. Then $\|a - P_m a\| \rightarrow 0$ for $m \rightarrow \infty$.

Proof.

$$\begin{aligned} \|a - P_m a\|^2 &= ((a, a) - 2((a, P_m a)) + ((P_m a, P_m a))) \\ &= \|a\|^2 - 2 \sum_{k=1}^m ((a, \omega_k)(a, \omega_k)_2) + \sum_{j,k=1}^m (a, \omega_j)_2 (a, \omega_j)_2 ((\omega_k, \omega_j)) \\ &= \|a\|^2 - 2 \sum_{k=1}^m (a, A\omega_k)_2 (a, \omega_k)_2 + \sum_{j,k=1}^m (a, \omega_k)_2 (a, \omega_j)_2 (\omega_k, A\omega_j)_2 \\ &= \|a\|^2 - 2 \sum_{k=1}^m \lambda_k (a, \omega_k)_2^2 + \sum_{j=1}^m \lambda_j (a, \omega_j)_2^2 = \|a\|^2 - \sum_{k=1}^m \lambda_k (a, \omega_k)_2^2. \end{aligned}$$

Since $\sum_{k=1}^m \lambda_k (a, \omega_k)_2^2 \xrightarrow{m \rightarrow \infty} \|a\|^2$ for $a \in N^1$ (see [12] p. 48) we obtain $\|a - P_m a\| \xrightarrow{m \rightarrow \infty} 0$, and this completes the proof.

1.4. We shall also consider the trilinear functional

$$(10) \quad b(u, v, w) := \sum_{i,j=1}^3 \int_{\Omega} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.$$

The integrals make sense in the case of $u, w \in L^4$ and $v \in N^1$. In virtue of the following particular case of Sobolev's imbedding theorem (see 5 p. 18)

$$H^1 \subset L^6$$

we deduce that the above functional is defined for $u, v, w \in N^1$.

We also have

$$(11) \quad |b(u, v, w)| \leq c_2 \|u\| \|v\| \|w\| \quad \text{for } u, v, w \in N^1,$$

where c_2 denotes a constant not depending on u, v, w (see [5] p. 95).

Finally let us remark that

$$(12) \quad b(u, v, w) = -b(u, w, v) \quad \text{for all } u, v, w \in N^1.$$

This relation is true if $u, v, w \in \mathcal{M}$ because

$$\begin{aligned} b(u, v, w) &= \int_{\Omega} \sum_{i,j=1}^3 u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx = - \int_{\Omega} \sum_{i,j=1}^3 v_j(x) \frac{\partial}{\partial x_i} [u_i(x) w_j(x)] dx \\ &= - \int_{\Omega} \sum_{i,j=1}^3 v_j(x) \frac{\partial u_i(x)}{\partial x_i} w_j(x) dx - \int_{\Omega} \sum_{i,j=1}^3 u_i(x) \frac{\partial w_j(x)}{\partial x_i} v_j(x) dx \\ &= - \int_{\Omega} \operatorname{div} u(x) \sum_{j=1}^3 v_j(x) w_j(x) dx - b(u, w, v). \end{aligned}$$

Since $\operatorname{div} u = 0$ in Ω for $u \in \mathcal{M}$ the first integral is equal to 0. We obtain (12) because \mathcal{M} is dense in N^1 and b is continuous.

LEMMA 3. Let $\{u_n\} \subset N^1$ be a sequence weakly convergent to u in N . Then $\lim_{n \rightarrow \infty} b(a, u_n, v) = b(a, u, v)$ for all $a, v \in N^1$.

Proof. Let a, v be any fixed elements of N^1 . In virtue of (11) there exists a continuous bilinear map $B: N^1 \times N^1 \rightarrow N^{-1}$ such that

$$(13) \quad \begin{aligned} b(a, v, u) &= (B(a, v), u)_2 \quad \text{for all } u \in N^1. \\ \lim_{n \rightarrow \infty} b(a, u_n, v) &= -\lim_{n \rightarrow \infty} b(a, v, u_n) = -\lim_{n \rightarrow \infty} (B(a, v), u_n)_2 = -(B(a, v), u)_2 \\ &= -b(a, v, u) = b(a, u, v) \end{aligned}$$

which was to be proved.

2. Generalized solutions of Oseen's equations

2.1. We shall consider Oseen's stationary equations

$$\begin{cases} -v \sum_{j=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \sum_{j=1}^3 a_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i}, & i = 1, 2, 3 \\ \operatorname{div} u = 0 & \text{in } \Omega \end{cases}$$

(here $a = (a_1, a_2, a_3)$, $f = (f_1, f_2, f_3)$ are given functions defined on Ω).

We add the condition

$$u(x) = 0 \quad \text{for all } x \in \partial\Omega.$$

This problem takes the form

$$(14) \quad v((u, v)) + b(a, u, v) = (f, v)_2 \quad \text{for all } v \in N^1,$$

where $u \in N^1$, $a \in N^1$, $f \in L^2$.

The above equation can be written in the form

$$(15) \quad vA_e u + B(a, u) = Pf \quad \text{in the sense of } N^{-1},$$

where P denotes the orthogonal projection of L^2 onto N .

DEFINITION 1. A generalized solution of Oseen's stationary equations with given $a \in N^1$ and $f \in L^2$ is an element $u \in N^1$ satisfying (15) (or (14)).

Obviously such solution satisfies the following energy equation

$$(16) \quad v \|u\|^2 = (f, u)_2.$$

We obtain this equation by putting $v = u$ in (13) and applying the condition

$$(17) \quad b(a, u, u) = 0 \quad \text{for all } u, a \in N^1$$

(it follows from (12)).

THEOREM 1. *The generalized solution of Oseen's stationary equations with given $f \in L^2$ and $a \in N^1$ always exists and is unique.*

PROOF. We show first that the generalized solution (if it exists) is unique. Let us suppose that $u_1, u_2 \in N^1$ are the solutions of (14). Then $u = u_1 - u_2$ satisfies the following equation

$$v((u, v)) + b(a, u, v) = 0 \quad \text{for all } v \in N^1.$$

This implies the energy equation

$$v \|u\|^2 = 0.$$

Consequently we obtain $u = 0$. Thus, if the solution exists then it is unique.

Let $\langle \omega_1, \dots, \omega_m \rangle$ denote the subspace of N^1 spanned by $\omega_1, \dots, \omega_m$ (here $\{\omega_j\}_{j=1}^\infty$ is the orthonormal basis of N introduced in 1.3.). We shall find an approximation solutions in the form $u_m = \sum_{k=1}^m c_k^{(m)} \omega_k$, where $c_k^{(m)} \in R$ for $k = 1, 2, \dots, m$, such that they satisfy the differential system

$$(18) \quad v((u_m, v)) + b(P_m a, u_m, v) = (f, v)_2 \quad \text{for } v \in \langle \omega_1, \dots, \omega_m \rangle$$

(P_m is orthogonal projection of N onto $\langle \omega_1, \dots, \omega_m \rangle$).

Let us put $v = \omega_k$ in (18). We obtain the following system of m -equations with unknown quantities $c_k^{(m)}, \dots, c_k^{(m)}$

$$(19) \quad v\lambda_k c_k^{(m)} + \sum_{j=1}^m c_j^{(m)} b(P_m a, \omega_j, \omega_k) = (f, \omega_k)_2 \quad (k = 1, \dots, m)$$

because $((u_m, \omega_k)) = c_k^{(m)} \lambda_k$ and $b(P_m a, u_m, \omega_k) = \sum_{j=1}^m c_j^{(m)} b(P_m a, \omega_j, \omega_k)$. If we denote $g_{jk} = b(P_m a, \omega_j, \omega_k)$ then we have $g_{jk} = -g_{kj}$ for $j, k = 1, \dots, m$. The matrix of the system (19) is as follows

$$M = \begin{bmatrix} v\lambda_1 & -g_{12} & -g_{13} & \dots & -g_{1m} \\ g_{12} & v\lambda_2 & -g_{23} & \dots & -g_{2m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ g_{1m} & g_{2m} & g_{3m} & \dots & v\lambda_m \end{bmatrix}.$$

Its quadratic form is $\psi(x) = \sum_{j=1}^m v\lambda_j x_j^2 > 0$ for all $x = (x_1, \dots, x_m) \neq 0$ in R^m . In virtue of Sylvester's theorem (see [10] p. 349) we deduce that there exists only one solution of (19). In this way we obtain the solutions $\{u_m\}_{m=1}^\infty$ of (18). Each such solution satisfies the energy equation

$$v \|u_m\|^2 = (f, u_m)_2.$$

It leads to

$$v \|u_m\|^2 \leq |f|_2 |u_m|_2 \leq c_1 \|u_m\| |f|_2 \quad (\text{from (4)}).$$

So that

$$(20) \quad \|u_m\| \leq \frac{c_1 |f|_2}{v} \quad \text{for } m = 1, 2, 3, \dots$$

In virtue of (20) there exists $\{u_{m_k}\} \subset \{u_m\}$ such that $\{u_{m_k}\}$ is weakly convergent (in N^1) to $u \in N^1$ (see [1] p. 147). We shall show that u is the generalized solution of Oseen's stationary equations. Obviously

$$((u_{m_k}, v)) \xrightarrow[k \rightarrow +\infty]{} ((u, v)) \quad \text{for all } v \in N^1.$$

From lemma 2 and lemma 3 we obtain

$$\begin{aligned} |b(P_{m_k} a, u_{m_k}, v) - b(a, u, v)| &\leq |b(a, u, v) - b(a, u_{m_k}, v)| + |b(a - P_{m_k} a, u_{m_k}, v)| \\ &\leq |b(a, u, v) - b(a, u_{m_k}, v)| + c_2 \|a - P_{m_k} a\| \|u_{m_k}\| \|v\| \\ &\leq |b(a, u, v) - b(a, u_{m_k}, v)| + \frac{c_1 c_2 |f|_2}{\nu} \|a - P_{m_k} a\| \|v\| \rightarrow 0 \text{ for } v \in N^1. \end{aligned}$$

Since $\{u_{m_k}\}$ satisfies (18) for all $v \in \langle \omega_1, \dots, \omega_{m_k} \rangle$ we can deduce that u verifies

$$v((u, v) + b(a, u, v)) = (f, v)_2 \quad \text{for all } v \in \bigcup_{k=1}^{\infty} \langle \omega_1, \dots, \omega_{m_k} \rangle.$$

The space $\bigcup_{k=1}^{\infty} \langle \omega_1, \omega_2, \dots, \omega_{m_k} \rangle = \bigcup_{j=1}^{\infty} \langle \omega_1, \dots, \omega_j \rangle$ is dense in N^1 so, that

$$v((u, v) + b(a, u, v)) = (f, v)_2 \quad \text{for all } v \in N^1$$

and this completes the proof of the theorem.

Remark. We shall show that the sequence $\{u_m\}$ is convergent to u in N^1 . For $u_m - P_m u$ we have

$$\begin{aligned} v((u, u_m - P_m u) + b(a, u, u_m - P_m u)) &= (f, u_m - P_m u)_2, \\ v((u_m, u_m - P_m u) + b(P_m a, u_m, u_m - P_m u)) &= (f, u_m - P_m u)_2. \end{aligned}$$

It leads to

$$\begin{aligned} |v((u - u_m, u - u_m))| &= |v((u - u_m, u - P_m u) + v((u - u_m, P_m u - u_m))| \\ &= |v((u, u - P_m u)) - v((u_m, u - P_m u)) + v((u, P_m u - u_m)) - v((u_m, P_m u - u_m))| \\ &\leq v(\|u\| + \frac{c_1 |f|_2}{\nu} \|u - P_m u\|) + |b(a, u, P_m u - u_m) - b(P_m a, u_m, P_m u - u_m)| \\ &\leq v\|u\| + \frac{c_1 |f|_2}{\nu} \|u - P_m u\| + |b(a, u, P_m u - u)| + |b(P_m a, u_m, P_m u - u)| + \\ &\quad + |b(a, u, u - u_m) - b(P_m a, u, u - u_m)| \\ &\leq v\|u\| + \frac{c_1 |f|_2}{\nu} \|u - P_m u\| + c_2 \|a\| \|u\| \|P_m u - u\| + c_2 \|P_m a\| \|u_m\| \|P_m u - u\| + \\ &\quad + c_2 \|a - P_m a\| \|u\| \|u - u_m\| \\ &\leq v\|u\| + \frac{c_1 |f|_2}{\nu} \|u - P_m u\| + c_2 \|a\| \|u\| \|P_m u - u\| + c_2 \frac{c_1 |f|_2}{\nu} \|a\| \|P_m u - u\| + \\ &\quad + c_2 \|a - P_m a\| \|u\| \left(\|u\| + \frac{c_1 |f|_2}{\nu} \right) \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

So, that

$$\lim_{m \rightarrow \infty} \|u - u_m\| = \lim_{m \rightarrow \infty} ((u - u_m, u - u_m))^{\frac{1}{2}} = 0.$$

2.2. For a given Banach space B (with the norm $\| \cdot \|_B$) we say that $f \in L^p(a, b; B)$ ($1 \leq p \leq +\infty$) if f is measurable and if

$$\|f\|_{L^p(a,b;B)} = \begin{cases} \left[\int_a^b \|f(t)\|_B^p dt \right]^{\frac{1}{p}} & \text{for } p < +\infty \\ \text{esssup}_{t \in (a,b)} \|f(t)\|_B & \text{for } p = +\infty \end{cases}$$

is finite.

If H is a Hilbert space then,

$$(f, g)_{L^2(a,b;H)} = \int_a^b (f(t), g(t))_H dt$$

is the scalar product in $L^2(a, b; H)$ and the space $L^2(a, b; H)$ is the Hilbert space (see [3] p. 236).

Obviously

$$\|f\|_{L^2(a,b;H)} = (f, f)_{L^2(a,b;H)}^{\frac{1}{2}} \quad \text{for } f \in L^2(a, b; H).$$

We consider Oseen's non-stationary equations

$$\begin{cases} \frac{\partial u_i}{\partial t} - \nu \sum_{j=1}^3 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \sum_{j=1}^3 a_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} & \text{for } i = 1, 2, 3 \\ \text{div}_x u := \sum_{j=1}^3 \frac{\partial u_j}{\partial x_j} = 0 & \text{in } \Omega, \end{cases}$$

where $a = (a_1, a_2, a_3)$, $f = (f_1, f_2, f_3)$ are given R^3 -valued functions of (t, x) belonging to the cylinder $[0, T] \times \Omega$.

We add the following initial-boundary conditions

$$\begin{aligned} u(t, x) &= 0 & \text{for } (t, x) \in [0, T] \times \Omega \\ u(0, x) &= u_0(x) & \text{for } x \in \partial\Omega \end{aligned}$$

(where $u_0(\cdot)$ is a given function such that $u_0(x) = 0$ for $x \in \partial\Omega$).

Using the following notations

$$u(t) = u(t, \cdot), \quad f(t) = f(t, \cdot), \quad a(t) = a(t, \cdot)$$

we can write the initial-boundary problem for Oseen's non-stationary equations in the form

$$(21) \quad u'(t) + \nu A_\epsilon u(t) + B(a(t), u(t)) = Pf(t),$$

$$(22) \quad u(0) = u_0.$$

DEFINITION 2. The *generalized solution of Oseen's non-stationary equations* (in $(0, T) \times \Omega$) with given functions $f \in L^2(0, T; L^2)$, $a \in L^2(0, T; N^1) \cap L^\infty(0, T; N^1)$ and initial value $u_0 \in N$ is a function $u(\cdot)$ such that

$$(23) \quad u \in L^2(0, T_1; N^1) \cap L^\infty(0, T_1; N) \quad \text{for every } T_1 \in (0, T)$$

u as N^{-1} -valued functions is an absolutely continuous function (i.e. there exists $u': (0, T) \rightarrow N^{-1}$ such that for every $v \in C_0^1(0, T; N^{-1})$ we have

$$-\int_0^T (u(t), v'(t))_{-1} dt = \int_0^T (u'(t), v(t))_{-1} dt$$

and verifies almost everywhere on $(0, T)$ (with respect to the Lebesgue measure) the differential equation (21) and initial condition (22), which are considered in N^{-1} .

Let $u(\cdot)$ be the generalized solution of Oseen's non-stationary equations. Then $u'(t) \in N^{-1}$ and verifies

$$(21') \quad (u'(t), v)_2 + v((u(t), v)) + b(a(t), u(t), v) = (f(t), v)_2$$

for all $v \in N^1$, a.e. in $(0, T)$ (the above equation is equivalent to (21)). In virtue of (4) and (11) we obtain

$$\begin{aligned} |(u'(t), v)_2| &\leq v \|u(t)\| \|v\| + c_2 \|a(t)\| \|u(t)\| \|v\| + |f(t)_2| v|_2 \\ &\leq (v \|u(t)\| + c_2 \|a(t)\| \|u(t)\| + c_1 |f(t)_2|) \|v\|. \end{aligned}$$

So that (from (8) and above inequality)

$$\begin{aligned} |u'(t)|_{-1} &\leq v \|u(t)\| + c_2 \|a(t)\| \|u(t)\| + c_1 |f(t)_2| \\ &\leq v \|u(t)\| + c_2 \|u(t)\| \operatorname{ess\,sup}_{t \in (0, T)} \|a(t)\| + c_1 |f(t)_2| \quad \text{a.e. in } (0, T). \end{aligned}$$

Thus

$$\int_0^T |u'(t)|_{-1}^2 dt < +\infty.$$

It shows that $u' \in L^2(0, T; N^{-1})$. The conditions $u \in L^2(0, T; N^1)$ and $u' \in L^2(0, T; N^{-1})$ imply that the function $(0, T) \ni t \rightarrow u(t) \in N$ is continuous (see [6] p. 33). This shows that the relation (22) makes sense i.e. $(u(t), v)_2 \xrightarrow{t \rightarrow 0} (u_0, v)_2$ for all $v \in N$. We also proved the following

LEMMA 4. *A generalized solution of Oseen's non-stationary equations (in the sense of above definition) is continuous from $(0, T)$ to N .*

Let us remark that the function $t \rightarrow (u(t), v)_2 \in R$ is absolutely continuous. For any fixed function $\varphi \in C_0^1(0, T; R)$ we have

$$\begin{aligned} \int_0^T (u(t), v)_2 \varphi'(t) dt &= \int_0^T (u(t), \varphi'(t)v)_2 dt = \int_0^T (u(t), I_e^{-1} v \varphi'(t))_{-1} dt \\ &= - \int_0^T (u'(t), I_e^{-1} v \varphi(t))_{-1} dt = - \int_0^T (u'(t), v)_2 \varphi(t) dt, \end{aligned}$$

where I_e is isometrical operator defined in 1.1.

Thus

$$(24) \quad \frac{d}{dt}(u(t), v)_2 = (u'(t), v)_2 \quad \text{for all } v \in N^1.$$

LEMMA 5. $u(\cdot)$ is a generalized solution of Oseen's non-stationary equations in the sense of above definition if and only if $u(\cdot)$ verifies (23) and satisfies the following equation

$$(25) \quad \int_0^T [-(u(t), v'(t))_2 + v((u(t), v(t))) + b(a(t), u(t), v(t))] dt \\ = (u_0, v(0))_2 + \int_0^T (f(t), v(t))_2 dt$$

for all functions $v(\cdot)$ verifying the following conditions

- (i) $v(\cdot) \in C([0, T]; N^1)$,
- (ii) $v(\cdot)$ is differentiable (in N) and its derivative $v'(\cdot)$ belongs to $L^2(0, T; N)$,
- (iii) $v(\cdot)$ has a compact support in $[0, T)$.

Proof. Let us suppose that $u(\cdot)$ is a generalized solution of Oseen's non-stationary equations in the sense of above definition. We shall denote

$$(26) \quad F(t) = (u(t), v(t))_2 - (u_0, v(0))_2 + \int_0^t -(f(s), v(s))_2 ds + \\ + \int_0^t [-(u(s), v'(s))_2 + v((u(s), v(s))) + b(a(s), u(s), v(s))] ds.$$

We assume that v is of the form $v(t) = \varphi(t)v$, here $v \in N^1$ and φ verifies the following conditions

- a) $\varphi \in C([0, T]; R)$,
- b) $\varphi' \in L^2(0, T; R)$,
- c) φ has a compact support in $[0, T)$.

In this particular case we can write (26) as follows

$$F(t) = \varphi(t)(u(t), v)_2 - \varphi(0)(u_0, v)_2 + \int_0^t -\varphi'(s)(u(s), v)_2 ds + \\ + \int_0^t [v((u(s), v)) + b(a(s), u(s), v) - (f(s), v)_2] \varphi(s) ds.$$

Let us remark (in virtue of (21') and (24)) that we have

$$\frac{d}{dt}(u(t), v)_2 + v((u(t), v)) + b(a(t), u(t), v) = (f(t), v)_2$$

for all $v \in N^1$, a.e. in $(0, T)$.

It leads to

$$\begin{aligned}
 F(t) &= \varphi(t)(u(t), v)_2 - \varphi(0)(u_0, v)_2 + \int_0^t -\varphi'(s)(u(s), v)_2 ds - \\
 &\quad - \int_0^t \varphi(s) \frac{d}{ds}(u(s), v)_2 ds \\
 &= \varphi(t)(u(t), v)_2 - \varphi(0)(u_0, v)_2 - \int_0^t \varphi'(s)(u(s), v)_2 ds - \varphi(s)(u(s), v)_2 \Big|_0^t + \\
 &\quad + \int_0^t \varphi'(s)(u(s), v)_2 ds = -\varphi(0)(u_0, v)_2 + \varphi(0)(u(0), v)_2.
 \end{aligned}$$

We have also (in virtue of (22)) $(u(0), v)_2 = (u_0, v)_2$ for all $v \in N^1$. It gives $F(t) = 0$. In the particular case of $t = T$ it leads to (25) for all $v(t) = \varphi(t)v$, where $v \in N^1$ and φ verifies a), b), c). We shall show that if $v(\cdot)$ satisfies (i)–(iii) then $v(\cdot)$ can be approached by the functions $\varphi(\cdot)v$ ($v \in N^1$, φ verifies a), b), c)). For every fixed $t \in (0, T)$ $v(t) \in N^1 \subset N$. It is obvious that

$$v(t) = \sum_{j=1}^{\infty} (v(t), \omega_j)_2 \omega_j, \quad v'(t) = \sum_{j=1}^{\infty} (v'(t), \omega_j)_2 \omega_j.$$

We shall denote

$$(v(t), \omega_j)_2 = \varphi_j(t), \quad (v'(t), \omega_j)_2 = \varphi'_j(t).$$

The functions $\varphi_j(\cdot)$ ($j = 1, 2, 3, \dots$) satisfy the conditions a), b), c). If $S_n(t) = \sum_{j=1}^n \varphi_j(t) \omega_j$, $S'_n(t) = \sum_{j=1}^n \varphi'_j(t) \omega_j$ then $v(t) = \lim_{n \rightarrow \infty} S_n(t)$, $v'(t) = \lim_{n \rightarrow \infty} S'_n(t)$ and (25) holds for $S_n(\cdot)$ ($n = 1, 2, \dots$). Applying Lebesgue's dominated convergence theorem we obtain that (25) is valid for all $v(\cdot)$ verifying (i)–(iii).

We can use this theorem because

$$\begin{aligned}
 |(u(t), S'_n(t))_2| &= |(u(t), \sum_{j=1}^n \varphi'_j(t) \omega_j)_2| \leq \frac{1}{2} \left[\sum_{j=1}^n |(v'(t), \omega_j)_2|^2 + \sum_{j=1}^n |(u(t), \omega_j)_2|^2 \right] \\
 &\leq \frac{1}{2} [\|v'(t)\|_2^2 + \|u(t)\|_2^2].
 \end{aligned}$$

In the same way we obtain

$$|((u(t), S_n(t)))| \leq \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|v(t)\|^2, \quad |(f(t), S_n(t))_2| \leq \frac{1}{2} \|v(t)\|_2^2 + \frac{1}{2} \|f(t)\|_2^2$$

and

$$|b(a(t), u(t), S_n(t))| \leq c_2 \|a(t)\| \|u(t)\| \sup_{t \in (0, T)} \|v(t)\|.$$

Thus, if $u(\cdot)$ is the generalized solution of Oseen's non-stationary equations, then $u(\cdot)$ verifies (23) and (25).

On the other hand, let $u(\cdot)$ be a function verifying (23) and (25) and let $v(\cdot)$ be a function from $(0, T)$ into N^{-1} of class C^1 with the compact support in $(0, T)$, which deriva-

