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DCXXIII PRACE MATEMATYCZNE, ZESZYT 23 1982

The one-dimensional Burgers' equation; existence, uniqueness and stability

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§ 1. Introduction. The existence, uniqueness and stability of the equation first investigated by J. M. Burgers [1] has been studied. This equation is derived from the theory of turbulent fluid motion and has properties similar to those of the Navier–Stokes equation, but is simpler to study. Investigations on the stability of this equation, or rather system of one differential-integral and a second partial differential equation, were reported recently [3]. It was shown that the solution $(U, v) = \left(\frac{P}{\nu}, 0\right)$ of (1) is locally stable for $\frac{P}{\nu} < \nu$, and the case $\frac{P}{\nu} = \nu$ was classificated as unstable.

We show that for $\frac{P}{\nu} < \nu$ the solution $\left(\frac{P}{\nu}, 0\right)$ is global exponential stable and for $\frac{P}{\nu} = \nu$ it is global asymptotic stable (for definitions see [2]).

First using the method of J. L. Lions [6] we show the existence and uniqueness of the solution of (1) in a weak sense, but our solution is sufficiently smooth to study the stability in a rather normal manner.

The one-dimensional Burgers' equation may be written in the following form:

$$\left| \begin{array}{ll} \dfrac{dU(t)}{dt} = P - \nu U(t) - \displaystyle\int_{\Omega} v^2(t, x)\, dx \quad \text{for } t>0, & (1.1) \\[2ex] \dfrac{\partial v(t, x)}{\partial t} = U(t) v(t, x) + \nu \dfrac{\partial^2 v(t, x)}{\partial x^2} - \dfrac{\partial}{\partial x}(v^2(t, x)) & (1.2) \end{array} \right.$$

in the set $Q := \{(t, x) \in R^2; t>0, x \in (0, \pi)\}$, $\Omega := (0, \pi)$, $U = U(t)\colon [0, \infty) \to R$, $v = v(t, x)\colon \bar{Q} \to R$, with the conditions

$$\left| \begin{array}{ll} U(0) = U_0 & \\ v(0, x) = \varphi(x) & \text{for } x \in \Omega, \\ v(t, 0) = v(t, \pi) = 0 & \text{for } t \geqslant 0. \end{array} \right. \qquad (2)$$

§ 2. Notation and definitions. The following real Banach spaces are considered (for details see [7] or [4]):

$C^m(\Omega)$ — the set of all real functions together with m-th order continuous derivatives (classical) in Ω,

$L^p(\Omega) = \{u\colon \Omega \to R;\ \|u\|_{L^p} = (\int_\Omega |u|^p dx)^{\frac{1}{p}} < \infty,\ u\text{-measurable}\}$,

$L^\infty(\Omega) = \{u\colon \Omega \to R;\ \|u\|_{L^\infty} = \operatorname{ess\,sup}|u| < \infty,\ u\text{-measurable}\}$,

$H^m(\Omega) = \{u\colon \Omega \to R;\ D^\alpha u \in L^2(\Omega),\ |\alpha| \leq m,\ u\text{-measurable}\}$, $D^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}$, where D^α is taken to be the derivative in the distributional sense ([4], [7], [8]).

By $H_0^1(\Omega)$ we mean the completion of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ ($\mathcal{D}(\Omega)$ as in [7]).

$$H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'.$$

The symbol "$f \in L^2(0, T; H_0^1(\Omega))$" denotes that the function $[0, T] \ni t \to f(t, \cdot) \in H_0^1(\Omega)$ belongs to $L^2(0, T)$. $\mathcal{L}(X, Y)$ denote the space of linear continuous operators from the Banach space X into the Banach space Y. The symbol $\langle\, ,\rangle$ denotes the real scalar product in $L^2(\Omega)$,

$\|\cdot\|_{H_0^1}$, $\|\cdot\|_{L^2}$ denotes the norm in $H_0^1(\Omega)$, $L^2(\Omega)$ respectively.

We also define certain other symbols

$$a(f, h) := \int_\Omega \frac{\partial f}{\partial x}\frac{\partial h}{\partial x} dx = \langle Af, h\rangle,$$

$$b(f, h, l) := \int_\Omega f \frac{\partial h}{\partial x} l\, dx = \langle g(f, h), l\rangle,$$

$$c(f, h, l) := \int_\Omega fhl\, dx = \langle C(f, h), l\rangle.$$

For any real fixed $T>0$ we define the weak solution of the problem (1), (2) as follows:

Definition 1. A pair of functions (U, v) is said to be *a weak solution* of the problem (1), (2) in $Q \cap \{t<T\}$ if U satisfies (1.1) in Caratheodory's sense (as in [5] or [2]);

$$U(t) = U_0 + \int_0^t \left(P - \nu U(\tau) - \int_\Omega v^2(\tau, x)\, dx\right) d\tau$$

for any $t \in [0, T]$, and U is absolutely continuous, and

(3) $$v \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$$

satisfies

(4) $$\langle v', w\rangle + \nu a(v, w) + 2b(v, v, w) = c(U, v, w)$$

for any function $w \in H_0^1(\Omega)$, $v' = \frac{\partial v}{\partial t}$ (distributional with the values in the Banach space [4], [7]),

(5) $$v(0, x) = \varphi(x) \equiv v_0(x) \in L^2(\Omega).$$

§ 3. We now prove certain lemmas which guarantee the correctness of the conditions (4) and (5).

LEMMA 1 (Wirtinger's inequality). *For any functions* $f(x) \in H_0^1(\Omega)$

(6) $$\|f\|^2_{L^2(\Omega)} \leqslant \|D_x f\|^2_{L^2(\Omega)}.$$

Proof. For any $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

where

$$a_n = \frac{2}{\pi} \int_{\Omega} f(x) \sin nx \, dx.$$

Differentiating the series with respect to x we get

$$f'(x) = \sum_{n=1}^{\infty} n a_n \cos nx$$

where the equality is satisfied in $L^2(\Omega)$. Now the inequality

$$\int_{\Omega} \sum_{n=1}^{\infty} a_n^2 \sin^2 nx \, dx \leqslant \int_{\Omega} \sum_{n=1}^{\infty} a_n^2 n^2 \cos^2 nx \, dx$$

and the fact that $a_n^2 \geqslant 0$ finishes the proof for the dense subset of $H_0^1(\Omega)$ and so for all $H_0^1(\Omega)$.

Let us note that the equality in (6) is possible only for the functions $f(x) = \text{const.} \sin x$, because $H_0^1(\Omega) \subset C^0(\bar{\Omega})$. The inclusion below is the consequence of the Sobolev embedding theorems (see [3], [6]);

$$H_0^1(\Omega) \subset C^0(\bar{\Omega}) \subset L^\infty(\Omega) \quad \text{for bounded } \Omega \subset R,$$

$$H_0^1(\Omega) \subset L_p(\Omega) \quad \text{for } p \geqslant 1.$$

Now we prove the following:

LEMMA 2. *For any functions* $f \in H_0^1(\Omega)$, $h, l \in L^2(\Omega)$

(7) $$\left|\int_{\Omega} fhl \, dx\right| \leqslant \|f\|^{\frac{1}{2}}_{H_0^1} \|f\|^{\frac{1}{2}}_{L^2} \|h\|_{L^2} \|l\|_{L^2}.$$

Proof. The following estimate is the consequence of the fact that $H_0^1(\Omega) \subset L^\infty(\Omega)$:

$$\left|\int_{\Omega} fhl \, dx\right| \leqslant \|f\|_{L^\infty} \|h\|_{L^2} \|l\|_{L^2}$$

For any function $f \in C^1(\bar{\Omega})$, $f(0) = f(\Pi) = 0$

$$|f(x)|^2 = f(x)^2 = \int_0^x D_s f(s) \cdot f(s) \, ds + \int_{\pi}^x D_s f(s) \cdot f(s) \, ds$$

$$\leqslant \left|\int_0^x D_s f(s) \cdot f(s) \, ds\right| + \left|\int_{\pi}^x D_s f(s) \cdot f(s) \, ds\right|$$

$$\leqslant \int_{\Omega} |D_s f(s) \cdot f(s)| \, ds \leqslant \|D_x f\|_{L^2} \|f\|_{L^2} \leqslant \|f\|_{H_0^1} \|f\|_{L^2}$$

and so (7) is satisfied for all $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$, and also for all $f \in H_0^1(\Omega)$.

For the functions $f, h \in H_0^1(\Omega)$ and $l \in L^2(\Omega)$ the following fact should be noted;

(8) $$|b(f,h,l)| = \left|\int_\Omega f\cdot D_x h\cdot l\,dx\right| \leqslant \|f\|_{H_0^1}^{\frac12}\|f\|_{L^2}^{\frac12}\|D_x h\|_{L^2}\|l\|_{L^2}.$$

Remark 1. For any function v satisfying Definition 1 we have

$$g(v,v)\in L^2(0,T;H^{-1}(\Omega)).$$

The following estimate $(w\in H_0^1(\Omega))$ ensure that this is so:

$$|b(v,v,w)| = \tfrac12|b(v,w,v)|\leqslant \tfrac12\|v\|_{H_0^1}^{\frac12}\|v\|_{L^2}^{\frac12}\|w\|_{H_0^1}\|v\|_{L^2}\leqslant\tfrac12\|v\|_{H_0^1}\|w\|_{H_0^1}\|v\|_{L^2}.$$

Remark 2. For any weak solution (U,v) the function

$$[0,T]\ni t\to\|v(t,x)\|_{L^2}^2$$

is measurable and belongs to $L^\infty(0,T)$. So U given by

(9) $$\frac{dU}{dt} = P-\nu U-\|v(t,x)\|_{L^2}^2$$

$$U(0) = U_0$$

has the derivative almost everywhere, and the derivative belongs to $L^\infty(0,T)$, which is the consequence of (9).

Lemma 3. *For any weak solution (U,v) of (1), (2) is $v'\in L^2(0,T;H^{-1}(\Omega))$.*

Proof. With our notation condition (4) may be written in the operator form:

(10) $$v' = -\nu A(v)-2g(v,v)+C(U,v).$$

Under our definition of the norm in $H_0^1(\Omega)$:

$$\|f\|_{H_0^1} := \sqrt{a(f,f)},$$

$$\|a(v,w)\|\leqslant\|v\|_{H_0^1}\|w\|_{H_0^1}$$

and the fact that $v\in L^2(0,T;H_0^1(\Omega))$ implies that

$$A(v)\in L^2(0,T;H^{-1}(\Omega)).$$

Also the following condition is satisfied

$$\left|\int_\Omega U(t)v(t,x)w(t,x)dx\right|\leqslant\|U\|_{C^0}\|v\|_{L^2}\|w\|_{L^2}\leqslant\|U\|_{C^0}\|v\|_{H_0^1}\|w\|_{H_0^1}$$

but $U\in C^0([0,T])$, so $C(U,v)\in L^2(0,T;H^{-1}(\Omega))$.
Hence Remark 1 finishes the proof of Lemma 3.

Lemma 4. *Any weak solution v before the change on the set of the null measure is continuous as the function*

$$[0,T]\ni t\to v(t,\cdot)\in L^2(\Omega).$$

Proof. The proof is the consequence of Theorem 3.1, 1.3 in [7], and the fact that

$$[H_0^1(\Omega), H^{-1}(\Omega)]_{\frac{1}{2}} = L^2(\Omega)$$

(the symbol $[\ ,\ ]_{\frac{1}{2}}$ as in [7]).

Remark 3. Lemma 4 makes condition (5) sensible. Moreover the function $U(t)$ satisfies (1.1) in the usual sense.

§ 4. Now we start to prove the existence of the weak solution of (1), (2), using the Bubnov–Galerkin method. The proof is parallel to the proof of existence for the Navier–Stokes equation in [6].

THEOREM 1. *There exists a weak solution of the problem* (1), (2) *in the sense of Definition* 1.

Proof. We are looking for the function $v(t, x)$ as the limit of the approximate solutions $v_m(t, x)$ of the form

(11) $$v_m(t, x) = \sum_{k=1}^{m} c_m^k(t)\, V_k(x)$$

where $\{V_k(x)\}_1^\infty$ is the "special" complete system in $H_0^1(\Omega)$. $\{V_k\}_1^\infty$ consists precisely of the own functions of the problem

$$\langle V_j, w\rangle_{H_0^1} := a(V_j, w) = \lambda_j \langle V_j, w\rangle$$

for any $w \in H_0^1(\Omega)$. The existence of such a base in H_0^1 for any dimension of the domain is a known fact (see [8], [6]).

The functions $c_m^k(t)$ and U_m are given by a system of $(m+1)$-ordinary differential equations:

(12.1) $$\frac{dU_m}{dt} = P - \nu U_m - \|v_m\|_{L^2}^2$$

(12.2) $$\langle v_m', V_l\rangle + \nu a(v_m, V_l) + 2b(v_m, v_m, V_l) = U_m \langle v_m, V_l\rangle, \quad l = 1, \dots, m$$

with the conditions

(12.3) $$\begin{aligned} &U_m(0) = U_{om}, && U_{om} \to U_0, \\ &v_m(0) = v_{om}, && v_{om} \to v_0 \quad \text{in } L^2(\Omega), \end{aligned}$$

where the convergence of v_{om} follows from (11).

The classical existence theorems ensure the existence of the solutions U_m, $\{c_m^l\}_1^m$ for any m. The functions c_m^k exist in the intervals $[0, t_m^k)$ and U_m exists in $[0, \min_k t_m^k)$.

Now we draw the "first a priori estimate", which guarantees the existence of $\{c_m^k\}_1^m$ and U_m in the whole interval $[0, T]$, and the fact that

U_m is bounded in $L^\infty(0, T)$,

v_m is bounded in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

Equation (12.1) is multiplied by U_m and integrated over $[0, t]$ for any $t \in [0, T]$, equations (12.2) are multiplied by c_m^l respectively, integrated over $[0, t]$ and summed over l

$$\frac{1}{2}\int_0^t \frac{d}{d\tau} U_m^2(\tau)\,d\tau + \nu \int_0^t U_m^2(\tau)\,d\tau = P\int_0^t U_m(\tau)\,d\tau - \int_0^t U_m(\tau)\,\|v_m(\tau)\|_{L^2}^2\,d\tau,$$

$$\frac{1}{2}\int_0^t \frac{d}{d\tau}\|v_m(\tau)\|_{L^2}^2\,d\tau + \nu\int_0^t \|v_m(\tau)\|_{H_0^1}^2\,d\tau = \int_0^t U_m(\tau)\,\|v_m(\tau)\|_{L^2}^2\,d\tau$$

summing these equalities by sides and using Young's inequality:

$$xy \leqslant \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2 \quad \text{for any } \varepsilon > 0$$

we have

$$(13)\quad \frac{1}{2}U_m^2(t) + \frac{1}{2}\|v_m(t)\|_{L^2}^2 + \nu\int_0^t [U_m^2(\tau) + \|v_m(\tau)\|_{H_0^1}^2]\,d\tau$$
$$\leqslant \frac{1}{2}U_m^2(0) + \frac{1}{2}\|v_{om}\|_{L^2}^2 + \frac{\varepsilon T}{2}P^2 + \frac{1}{2\varepsilon}\int_0^t U_m^2(\tau)\,d\tau$$

setting $\nu - \frac{1}{2\varepsilon} > 0$ we obtain that

U_m are bounded in $L^\infty(0, T)$,

v_m are bounded in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

In the next part of the proof we show that v_m' are bounded in $L^2(0, T; H^{-1}(\Omega))$. Denoting as P_m the projection

$$L^2(\Omega) \to \lim(V_1, \ldots, V_m)$$

such that

$$P_m h = \sum_{l=1}^m \langle h, V_l\rangle V_l,$$

multiplying (12.2) by V_l and summing over l from 1 to m we get

$$(14)\quad \sum_l^m \langle v_m', V_l\rangle V_l = -\nu \sum_l^m \int_\Omega \frac{\partial v_m}{\partial x}\frac{\partial V_l}{\partial x}\,dx\,V_l$$
$$-2\sum_l^m \int_\Omega v_m \frac{\partial v_m}{\partial x} V_l\,dx\,V_l + U_m \sum_l^m \langle v_m, V_l\rangle V_l.$$

Note that $P_m = P_m'$, and

$$\sum_l^m \langle v_m', V_l\rangle V_l = v_m'$$

because

$$v_m' = \sum_k^m \frac{dc_m^k}{dt} V_k \in \lim(V_1, \ldots, V_m).$$

Multiplying (14) by any function $w \in H_0^1(\Omega)$ and integrating over Ω we obtain the operator equality

(15) $$v_m' = -2P_m(g(v_m)) - \nu P_m A v_m + P_m C(U_m, v_m).$$

Taking in the definition of V_j; $w = V_j$ we obtain that

$$\lambda_j = \|V_j\|_{H_0^1}^2 : \|V_j\|_{L^2}^2$$

and so by Lemma 1 it follows that $\lambda_j \geqslant 1$. Now

$$\left\| \sum_{j=1}^m \langle V_j, w\rangle V_j \right\|_{H_0^1} \leqslant \left\| \sum_{j=1}^m \frac{1}{\lambda_j} \langle w, V_j\rangle_{H_0^1} \cdot V_j \right\|_{H_0^1} \leqslant \|w\|_{H_0^1}$$

because $w \in H_0^1(\Omega)$, and $\frac{1}{\lambda_j} \leqslant 1$ for any j. So we obtain $\|P_m\|_{\mathscr{L}(H_0^1, H_0^1)} \leqslant 1$ and by $P_m = P_m'$

(16) $$\|P_m\|_{\mathscr{L}(H^{-1}, H^{-1})} \leqslant 1.$$

Just as in Remark 1 and Lemma 3 $g(v_m)$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, also Av_m and $C(U_m, v_m)$ are bounded in $L^2(0, T; H^{-1}(\Omega))$, and so (15) and (16) imply that v_m' is bounded in $L^2(0, T; H^{-1}(\Omega))$.

Now we make use of the "theorem of compactness" [6], Theorem 5.1, 1.5, with

(17) $$B_0 = H_0^1(\Omega) \subset B = L^2(\Omega) \subset B_1 = H^{-1}(\Omega).$$

In our situation Theorem 5.1 laid down that for any sequence v_m, such that the following sequences are bounded

$$v_m \in L^2(0, T; H_0^1(\Omega)) \text{ and } v_m' \in L^2(0, T; H^{-1}(\Omega))$$

a subsequence v_μ may be taken such that v_μ is convergent in $L^2(0, T; L^2(\Omega)) = L^2(Q)$.

Now using the well-known properties of bounded sets in reflexive spaces (see [4]) we may take a subsequence of v_μ such that

(18) $$v_\eta \to v \quad \text{weak in } L^2(0, T; H_0^1(\Omega)),$$

(19) $$v_\eta \to v \quad \text{weak star in } L^\infty(0, T; L^2(\Omega)),$$

(20) $$v_\eta \to v \quad \text{in } L^2(Q), \text{ almost everywhere in } Q,$$

(21) $$v_\eta \to v' \quad \text{weak in } L^2(0, T; H^{-1}(\Omega)).$$

By (18) and (20) and Lemma 1.2, 1,2 in [6]

$$v_{0\eta} \to v_0 \quad \text{weak in } H^{-1}(\Omega), \text{ but from (12.3)}$$

$$v_{0\eta} \to v_0 \quad \text{in } L^2(\Omega), \text{ and so}$$

$$v(0, x) = v_0(x) \quad \text{in } L^2(\Omega).$$

It follows from Lemma 2 that for $f \in C^1(\overline{\Omega})$, $f(0) = f(\pi) = 0$

(22) $$\|f(x)\|_{L^4}^4 \leqslant \|f\|_{H^1_0}^2 \|f\|_{L^2}^2 \int_\Omega 1\,dy = \pi \|f\|_{H_0^1}^2 \|f\|_{L^2}^2.$$

Using (22) and (3) we obtain that v_η^2 are bounded in $L^2(0, T; L^2(\Omega)) = L^2(Q)$ and so v_η^2 are weak convergent in $L^2(Q)$. But v_η from (20) is almost everywhere convergent in Q to v, so

(23) $$v_\eta^2 \to v^2 \quad \text{weak in } L^2(Q).$$

Finally from the condition

$$\int_0^T b(v_\eta, v_\eta, V_j)\psi\,dt = -\tfrac{1}{2}\int_0^t b(v_\eta, V_j, v_\eta)\psi\,dt$$

for any $\psi \in L^2(0, T)$, and by (23) it follows that

$$b(v_\eta, v_\eta, V_j) \to b(v, v, V_j) \quad \text{weak in } L^2(0, T).$$

Now (18) implies that

$$a(v_\eta, V_j) \to a(v, V_j) \quad \text{weak in } L^2(0, T)$$

and (21) implies that

$$\langle v_\eta', V_j\rangle \to \langle v', V_j\rangle \quad \text{in } \mathscr{D}'(0, T) \text{ (see [6])}.$$

It remains to study the component $U_m\langle v_\eta, V_l\rangle$. From (20) it follows that

$$\langle v_\eta, V_l\rangle \to \langle v, V_l\rangle.$$

We show now that U_m converges uniformly to U in $[0, T]$.

Remark 4. U_m converges uniformly to U in $[0, T]$.
The condition (20) implies that $\|v_\eta\|_{L^2(Q)}^2 \to \|v\|_{L^2(Q)}^2$. Let U_m satisfy (12,1)

$$U_m(t) = U_m(0) + \int_0^t [P - \nu U_m - \|v_m\|_{L^2}^2]\,d\tau$$

Then the difference $U_m - U_n$ satisfies

(24) $$|U_m(t) - U_n(t)| \leqslant [|U_m(0) - U_n(0)| + \varepsilon_{mn}]\exp(\nu t),$$

$$\text{where } \varepsilon_{mn} := |\,\|v_m\|_{L^2(Q)}^2 - \|v_n\|_{L^2(Q)}^2| \to 0.$$

All the facts previously studied allow us to pass with η to infinity in (12), and the fact that $\{V_l\}_1^\infty$ is the complete system in $H_0^1(\Omega)$ shows that there exists a weak solution of (1), (2).

Now we show that under the "smoothness" initial conditions our solution exists in a better sense.

THEOREM 2. *If* $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, *then for the weak solution of* (1), (2) *we have*

$$v' \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)). \tag{25}$$

Proof. Let now $\{V_k\}_1^\infty$ denote the base in $H_0^1(\Omega) \cap H^2(\Omega)$. Differentiating (12.2) with respect to t we obtain:

$$\langle v_m'', V_l\rangle + \nu a(v_m', V_l) + 2b(v_m', v_m, V_l) + 2b(v_m, v_m', V_l) = U_m'\langle v_m, V_l\rangle + U_m\langle v_m', V_l\rangle$$

$$v_{0m} \to v_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega).$$

Multiplying by $c_m^{l'}$ and summing over l from 1 to m:

$$\langle v_m'', v_m'\rangle + \nu a(v_m', v_m') + 2b(v_m', v_m, v_m') + \\ + 2b(v_m, v_m', v_m') = U_m'\langle v_m, v_m'\rangle + U_m\langle v_m', v_m'\rangle. \tag{26}$$

The following estimates are now used:

$$2b(v_m', v_m, v_m') + 2b(v_m, v_m', v_m') = -4b(v_m', v_m', v_m) + \\ + 2b(v_m', v_m', v_m) = -2b(v_m', v_m', v_m)$$

$$|-2b(v_m', v_m', v_m)| \leq 2\|v_m'\|_{H_0^1}^{\frac{3}{2}} \|v_m'\|_{L^2}^{\frac{1}{2}} \|v_m\|_{L^2}$$

$$\leq \varepsilon_1 \|v_m'\|_{H_0^1}^2 \|v_m\|_{L^2}^2 + \frac{\varepsilon_2}{2\varepsilon_1} |v_m'|_{L^2}^2 + \frac{1}{2\varepsilon_1 \varepsilon_2} \|v_m'\|_{H_0^1}^2$$

$$|U_m'\langle v_m, v_m'\rangle| \leq \tfrac{1}{2}|U_m'|^2 \|v_m\|_{L^2}^2 + \tfrac{1}{2}\|v_m'\|_{L^2}^2$$

and $U_m' \in L^\infty(0,T)$,

$$|U_m\langle v_m', v_m'\rangle| \leq |U_m| \, \|v_m'\|_{L^2}^2.$$

Choosing constants ε_1 and ε_2 such that

$$\beta := \nu - \text{ess sup}\left(\frac{1}{2\varepsilon_1 \varepsilon_2} - \varepsilon_1 \|v_m\|_{L^2}^2\right) > 0$$

and integrating (26) over $[0,t]$ we get

$$\frac{1}{2}\|v_m'(t)\|_{L^2}^2 + \beta \int^t \|v_m'\|_{H_0^1}^2 d\tau \leq \frac{1}{2}\|v_m'(0)\|_{L^2}^2 + \frac{1}{2}\int_0^t |U_m'| \, \|v_m\|_{L^2}^2 d\tau + \\ + \int_0^t \|v_m'\|_{L^2}^2 \left[\frac{\varepsilon_2}{2\varepsilon_1} + \frac{1}{2} + |U_m|\right] d\tau \leq \frac{1}{2}\left[c_1 + c_2 \int \|v_m'\|_{L^2}^2 d\tau\right] \tag{27}$$

so by Gronwalls' inequality

$$\|v_m'(t)\|_{L^2}^2 \leq c_1 \exp(c_2 T) \tag{28}$$

from which v'_m is bounded in $L^\infty(0, T; L^2(\Omega))$, and by (27) v'_m is bounded in $L^2(0, T; H_0^1(\Omega))$. We must show only that $\|v'_m(0)\|^2_{L^2}$ in (27) is bounded. We multiply (12.2) by $c_m^{l'}(0)$ and sum over l from 1 to m, with $t = 0$

$$(29) \quad \|v'_m(0)\|^2_{L^2} \leqslant \nu \left| \int_\Omega \frac{\partial^2 v_{0m}}{\partial x^2} v'_m(0)\, dx \right| + \\ + 2\|v'_m(0)\|_{L^2} \|v_{0m}\|^{\frac{3}{2}}_{H_0^1} \|v_{0m}\|^{\frac{1}{2}}_{L^2} + |U_m| \, \|v_{0m}\|_{L^2} \times \\ \times \|v'_m(0)\|_{L^2} \leqslant c_3 \|v'_m(0)\|_{L^2}$$

because $v_{0m} \to v_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$. We have also used the following estimate for the functions v_{0m} converging to v_0 in $H_0^1(\Omega) \cap H^2(\Omega)$:

$$\left\| \frac{\partial^2}{\partial x^2} v_{0m} \right\| \leqslant \text{const}.$$

and so

$$\nu |a(v_{0m}, v'_m(0))| = \nu \left| \int_\Omega \frac{\partial^2}{\partial x^2} v_{0m} \cdot v'_m(0)\, dx \right| \leqslant \nu \,\text{const.}\, \|v'_m(0)\|_{L^2}.$$

From (29) it follows that $\|v'_m(0)\|^2_{L^2}$ is bounded.

To finish the proof of Theorem 2 it remains to note that $v_m \to v$ weak in $L^2(0, T; H_0^1(\Omega))$, and so we may pass to the limit in (12.2).

Remark 5. From (3) and (25) using Sobolev's theorem it follows that

$$(30) \qquad v \in C^0([0, T]; H_0^1(\Omega)),$$

so the initial condition for v is satisfied in the sense of (30).

§ **5.** Now we show the uniqueness of the weak solution (U, v) of the problem (1), (2).

THEOREM 3. *There exists at most one solution of the problem* (1), (2) *in the sense of Definition* 1.

Proof. Let (U_1, v_1) and (U_2, v_2) be two different solutions of (1), (2). We put

$$U = U_1 - U_2, \quad (U(0) = 0),$$

$$v = v_1 - v_2, \quad (v(0, x) = 0 \text{ in } L^2(\Omega)).$$

Then it is satisfied that

$$(31.1) \quad U' = -\nu U - \|v_1\|^2_{L^2} + \|v_2\|^2_{L^2} \quad \text{(almost everywhere, when } U' \text{ exists)},$$

$$(31.2) \quad \langle v', V\rangle = U_1\langle v, V\rangle + U\langle v_2, V\rangle - \nu a(v, V) - 2b(v_1, v, V) - 2b(v, v_2, V).$$

Multiplying (31.1) by U and putting in (31.2) $V=v$ (v belongs to $H_0^1(\Omega)$ for almost all $t\in[0,T]$) we get:

$$\tfrac{1}{2}(U^2)' = -\nu U^2 - U\int_\Omega v(v_1+v_2)\,dx$$

$$\langle v', v\rangle = -\nu\|v\|^2_{H_0^1} + U_1\|v\|^2_{L^2} + U\langle v_2, v\rangle - 2b(v_1, v, v) - 2b(v, v_2, v)$$

now by

$$2b(v, v_2, v) = -4b(v, v, v_2)$$

we have

$$\frac{1}{2}(U^2)' + \nu U^2 \leqslant \frac{c_1}{2}U^2\|v_1+v_2\|^2_{L^2} + \frac{1}{2c_1}\|v\|^2_{L^2} \tag{32.1}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2} + \nu\|v\|^2_{H_0^1} \leqslant c_2\|v\|^2_{L^2} + \frac{c_3}{2}\|v_2\|^2_{L^2}U^2 + \frac{1}{2c_3}\|v\|^2_{L^2} + 2|b(v_1, v, v)| + 4|b(v_2, v, v)| \tag{32.2}$$

now

$$|b(v_i, v, v)| \leqslant \|v\|^{\frac{1}{2}}_{L^2}\|v\|^{\frac{3}{2}}_{H_0^1}\|v_i\|_{L^2}$$

$$\leqslant \frac{\varepsilon_i}{2}\|v\|^2_{H_0^1}\|v_i\|^2_{L^2} + \frac{\varepsilon_{ii}}{4\varepsilon_i}\|v\|^2_{L^2} + \frac{1}{4\varepsilon_{ii}\varepsilon_i}\|v_i\|^2_{H_0^1}, \qquad i = 1, 2.$$

Choosing the constants in such a way that:

$$\alpha_1 = \nu - \text{ess sup}\left(\frac{c_1}{2}\|v_1+v_2\|^2_{L^2} - \frac{c_3}{2}\|v_2\|^2_{L^2}\right) > 0$$

$$\alpha_2 = \nu - \text{ess sup}\left(\varepsilon_1\|v_1\|^2_{L^2} - 2\varepsilon_3\|v_2\|^2_{L^2} - \frac{1}{2\varepsilon_{11}\varepsilon_1} - \frac{1}{\varepsilon_{22}\varepsilon_2}\right) > 0$$

summing (32) together, integrating over $[0, t]$ and using the fact that $U(0)=0$, $v(0, x)=0$ in $L^2(\Omega)$ we obtain

$$\tfrac{1}{2}U^2(t) + \tfrac{1}{2}\|v(t)\|v^2_{L^2} \leqslant \tfrac{1}{2}U^2(t) + \tfrac{1}{2}\|v(t)\|^2_{L^2} + \alpha_1\int_0^t U^2(\tau)\,d\tau + \alpha_2\int_0^t\|v(\tau)\|^2_{H_0^1}\,d\tau \leqslant \text{const.}\int_0^t\|v(\tau)\|^2_{L^2}\,d\tau$$

and so by Gronwall's inequality

$$U^2(t) + \|v(t, x)\|^2_{L^2} = 0 \quad \text{in } [0, T],$$

which finishes the proof of Theorem 3.

§ 6. Finally we shall study the stability of the solution $\left(\frac{P}{\nu}, 0\right)$ of (1), (2).

All the definitions for this part will be found in [2].

As the solution we mean the functions U, v in the weak sense (see Remark 3). $U(t)$ satisfies (1.1) in the classical sense and the function $z(t) := \|v(t,x)\|^2_{L^2}$ is continuous. The regularity allowed us to study the following system obtained from (4) and (1.1):

$$(33)\qquad \left|\begin{array}{l} \dfrac{dU}{dt} = P - \nu U - z \\ \frac{1}{2}z(T_2) - \frac{1}{2}z(T_1) = \int\limits_{T_1}^{T_2} U(\tau)z(\tau)\,d\tau - \nu \int\limits_{T_1}^{T_2} \|v(\tau)\|^2_{H^1_0}\,d\tau \end{array}\right.$$

with the conditions

$$U(0) = U_0, \qquad z(0) = \|v_0\|^2_{L^2} = z_0\,,$$

where $0 \leqslant T_1 \leqslant T_2$-arbitrary.

Our purpose is to show the global exponential stability of the solution $\left(\frac{P}{\nu}, 0\right)$ of (33) when $\frac{P}{\nu} < \nu$. Using the transformation $W(t) = U(t) - \frac{P}{\nu}$ we arrive at the problem of the stability of the (0, 0) solution for

$$(34.1)\qquad \frac{dW}{dt} = -\nu W - z$$

$$(34.2)\qquad \frac{1}{2}\big(z(T_2) - z(T_1)\big) = \int\limits_{T_1}^{T_2} \left(W(\tau) + \frac{P}{\nu}\right) z(\tau)\,d\tau - \nu \int\limits_{T_1}^{T_2} \|v(\tau)\|^2_{H^1_0}\,d\tau$$

$$W(0) = U_0 - \frac{P}{\nu}, \qquad z(0) = z_0\,.$$

We start with the following;

Lemma 5. *When $W(t) > 0$, then W is strictly decreasing.*

Proof. The proof is the consequence of (34.1) and the fact that $z(t) \geqslant 0$.

By Lemma 5 it follows that if the function W exists in all real axes, then $\limsup_{t\to\infty} W(t) \leqslant 0$. Also when for some $t_0 \geqslant 0$ is $W(t_0) = 0$, then $W(t) \leqslant 0$ for $t \geqslant t_0$.

Now we show that the solutions W and z are bounded on $[0, \infty)$ when $\frac{P}{\nu} \leqslant \nu$, which implies that

$$U \in C^1([0,\infty)) \qquad \text{and is bounded,}$$

$$v \in C^0([0,\infty); L^2(\Omega)) \qquad \text{and is bounded.}$$

Theorem 4. *When $\frac{P}{\nu} \leqslant \nu$ then the solution (W, z) of (34) is bounded independently on t.*