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The classical solution of the one-dimensional Burgers' equation

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§ 1. Introduction. A continuation of the author's investigations [1] is given. The classical existence of the solution of Burgers' problem, its boundedness and some remarks concerning the structure of non-stability for $\frac{P}{v} > v$ are presented.

The following system is studied:

$$\left| \begin{array}{ll} \dfrac{dU(t)}{dt} = P - vU(t) - \displaystyle\int_{\Omega} v^2(t, x)\,dx & (1.1) \\[2ex] \dfrac{\partial v(t, x)}{\partial t} = U(t)v(t, x) + v\dfrac{\partial^2 v(t, x)}{\partial x^2} - \dfrac{\partial}{\partial x}(v^2(t, x)), & (1.2) \end{array} \right.$$

where $x \in \Omega = (0, \pi)$, $t \in (0, T)$, $Q = \Omega \times (0, T)$, with the conditions

$$U(0) = U_0, \quad v(0, x) = \varphi(x), \quad v(t, 0) = v(t, \pi) = 0\,. \tag{1.3}$$

The symbols defined in [1] remains unchanged. The following new spaces are used:

$C^{2+\alpha, 1+\frac{\alpha}{2}}(Q)$ — the Banach space of all functions continuous in $\bar{Q}$ together with the derivatives v_{xx}, v_t, with the finite norm

$$\|f\|_{\bar{Q}}^{2+\alpha, 1+\frac{\alpha}{2}} = \sum_{\substack{k+2l \leqslant 2 \\ k, l \geqslant 0}} [\|D_x^k f\|_{\bar{Q}}^{\alpha} + \|D_t^l f\|_{\bar{Q}}^{\frac{\alpha}{2}}], \quad \alpha \in (0, 1),$$

where

$$\|D_x^k f\|_{\bar{Q}}^{\alpha} = \sup_{\bar{Q}} |D_x^k f| + \sup_{P, R \in \bar{Q}} \frac{|D_x^k f(P) - D_x^k f(R)|}{d(P, R)}$$

$$P = (t, x), \quad R = (\bar{t}, \bar{x}), \quad d(P, R) = |x - \bar{x}|^{\alpha} + |t - \bar{t}|^{\frac{\alpha}{2}},$$

$$\|D_t^l f\|_{\bar{Q}}^{\frac{\alpha}{2}} = \sup_{\bar{Q}} |D_t^l f| + \sup_{P, R \in \bar{Q}} \frac{|D_t^l f(P) - D_t^l f(R)|}{d(P, R)}$$

$C^{2+\alpha}(\bar{\Omega})$ — the analogous space for functions $f(x)$ defined for $x \in \bar{Q}$,

$C^{2,1}(Q \cup Q_T)$ — the set of continuous in $\bar{Q}$ functions, possesses continuous in $Q \cup Q_T$ derivatives v_t, v_x, v_{xx} $(Q_T = \Omega \times \{t \leqslant T\})$.

§ 2. In [1] the following theorem is proved:

THEOREM 1. *If the initial function* $\varphi(x) \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exists a unique "weak solution"* (U, v) *(defined in* [1]*), and* $U(t)$ *satisfies* (1.1) *in a classical sense,* $v(t, x) \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ *and the distributional derivative* $v' = D_t v$ *satisfies* $v'(t, x) \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. *Moreover the function* $[0, T] \ni t \to v(t, \cdot) \in L^2(\Omega)$ *is continuous.*

Now the following will be shown

LEMMA 1. *The solution* v *defined in Theorem* 1 *is continuous in* $\bar{Q}$.

Proof. From the condition $v \in L^2(0, T; H_0^1(\Omega))$, $v' \in L^2(0, T; H_0^1(\Omega))$ and Theorem 3.1, 1, 3 of [3] the function $v(t, x)$ is continuous as $[0, T] \ni t \to v(t, \cdot) \in H_0^1(\Omega)$. Next from the Sobolev embedding theorem $H_0^1(\Omega) \subset C^0(\bar{\Omega})$.

The following difference must be estimated

$$|v(t, x) - v(t_0, x_0)| \leqslant |v(t, x) - v(t, x_0)| + |v(t, x_0) - v(t_0, x_0)| .$$

The first component converges to 0 for $x \to x_0$ because $H_0^1(\Omega) \subset C^0(\bar{\Omega})$, and also the second component for $t \to t_0$.

Hence the solution $v(t, x)$ satisfies the initial and boundary conditions in the classical sense.

Now the following is considered

THEOREM 2. *If* $\varphi(x) \in C^{2+\alpha}(\bar{\Omega})$ *and the condition below is satisfied*

$$\nu\varphi_{xx}(x) - 2\varphi(x)\varphi_x(x) + U_0\varphi(x)|_{x=0,\Pi} = 0 , \tag{2}$$

$$\varphi(0) = \varphi(\pi) = 0$$

then there exists a solution $v(t, x) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q})$, $U(t) \in C^{2+\frac{\alpha}{2}}([0, T])$.

The proof follows directly from Theorem 5.2, VI, 5 of [4], which for our assumptions will be formulated as follows: the following problem is studied

$$\frac{\partial w}{\partial t} - a_{11}\frac{\partial^2 w}{\partial x^2} + a(t, x, w, w_x) = 0 \tag{3}$$

$a_{11} = \text{const.} > 0$, with the condition

$$w/_{\Gamma_T} = \psi(t, x) , \tag{4}$$

where $\Gamma_T := \{\Omega \times \{t = 0\}\} \cup \{\partial\Omega \times [0, T]\}$.

PROPOSITION 1. *Let the following conditions be satisfied:*

a) *for* $(t, x) \in \bar{Q}$ *and any* w

$$-a(t, x, w, 0)w \leqslant |w|\Phi(|w|), \qquad \int_1^\infty \frac{d\tau}{\Phi(\tau)} = \infty ,$$

where Φ *denotes a non-decreasing positive function,*

b') *for* $(t, x) \in \bar{Q}$, $|w| \leqslant M$

$$|a(t, x, w, p)| \leqslant \mu(|w|)\cdot(1+|p|)^2 ,$$

where $\mu(t)$ *is a positive non-decreasing continuous function of* $t \geqslant 0$,

c) *for* $(t, x) \in \bar{Q}$, $|w| \leqslant M$, $|p| \leqslant M_1$

$a(t, x, w, p)$ *is Hölder continuous in t with the exponent* $\frac{\alpha}{2}$ *and in x, w, p with the exponent* α,

d) $\psi(t, x) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q})$ *and*

$$[\psi_t(0, x) - a_{11}\psi_{xx}(0, x) + a(0, x; \psi(0, x), \psi_x(0, x))]|_{x=0,\pi} = 0 .$$

Then the problem (3)–(4) *has a unique solution in* $C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q})$.

For our case

$$a(t, x, v, p) = -U(t)v + 2vp ,$$

where $U(t)$ belongs to $C^1([0, T])$ (from Theorem 1). The assumptions of Proposition 1 are satisfied. In d) we put

$$\psi(t, x) \equiv \varphi(x) \quad \text{for all } t \in [0, T], \quad \text{and hence } \psi_t(0, x) = 0 .$$

Remark 1. If v has the properties given by Theorem 2 then $U(t) \in C^{2+\frac{\alpha}{2}}([0, T])$. This is the consequence of (1.1);

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_\Omega v^2(t, x)dx$$

and

$$\int_\Omega v^2(t, x)dx \in C^{1+\frac{\alpha}{2}}([0, T]) .$$

Remark 2. In Burgers' problem the physical condition $v(t, 0) = v(t, \pi) = 0$ must be satisfied. It should be noted that (2) is then satisfied when

$$\varphi(0) = \varphi(\pi) = \varphi_{xx}(0) = \varphi_{xx}(\pi) = 0 .$$

It will be attempted to show that the solution given in Theorem 1 is the classical one, i.e. that $v \in C^{2,1}(Q \cup Q_T)$, and satisfies (1.2) in $Q \cup Q_T$. The solution will be approximated with those given in Theorem 2.

We start with the following:

Lemma 2. *If* $(U_1(t), v_1(t,x))$ *and* $(U_2(t), v_2(t,x))$ *are the two classical solutions of Burgers' problem, with the initial functions* $(U_1(0), \varphi_1(x))$ *and* $(U_2(0), \varphi_2(x))$ *respectively, and moreover if the derivatives* $(v_1)_x$, $(v_2)_x$ *are bounded in* $Q \cup Q_T$, *then*

$$\max_{\bar{Q}}|v_1(t,x)-v_2(t,x)| \leqslant \max\left\{\max_{\bar{\Omega}}|\varphi_2(x)-\varphi_2(x)|\,;\right. \tag{5}$$

$$\left.\frac{|U_1(0)-U_2(0)|+c_1\sqrt{|U_1^2(0)-U_2^2(0)|}+c_2\,\|\varphi_1(x)-\varphi_2(x)\|_{L^2(\Omega)}}{\left[[\lambda-2\max\left(-\dfrac{\partial v_2}{\partial x}\right)-\max(U_1)]\cdot\exp(-\nu T)\right]}\right\}\exp(\lambda T)$$

where c_1 *and* c_2 *are positive constants, depending only on* T, ν, $\max|v_i|$; $i = 1, 2$, *and* λ *satisfies*

$$\lambda > 2\max\left|\frac{\partial v_2}{\partial x}\right| + \max|U_1|\,.$$

Proof. Denoting $W(t,x) = v_1(t,x)-v_2(t,x)$ and defining the function

$$V(t,x) = W(t,x)\exp(-\lambda t)$$

it is obtained that $V(t,x)$ satisfies

$$\frac{\partial V}{\partial t} = -\lambda V+\nu\frac{\partial^2 V}{\partial x^2}+\left(U_1 v_1-U_2 v_2-2v_1\frac{\partial v_1}{\partial x}+2v_2\frac{\partial v_2}{\partial x}\right)\exp(-\lambda t) \tag{6}$$

in the classical sense. There are three possibilities;

e) *$V(t,x)\leqslant 0$ in $\bar{Q}$,*

f) *V admits the largest positive value in Γ_T,*

g) *the largest positive value is admitted in $\bar{Q}\setminus\Gamma_T$.*

In case e) $V(t,x)\leqslant 0$, in f) $\max_{\bar{Q}} V(t,x) \leqslant \max_{\Gamma_T} V(t,x) = \max_{\bar{\Omega}}(\varphi_1(x)-\varphi_2(x))$. It remains to study the g) case. Let V have the maximal positive calue in $(t_0, x_0) \in Q \cup Q_T$. We have

$$V_t(t_0, x_0) \geqslant 0, \qquad V_x(t_0, x_0) = 0, \qquad V_{xx}(t_0, x_0) \leqslant 0$$

and from (6) it follows that

$$\frac{\partial V}{\partial t} = -\lambda V+\nu\frac{\partial^2 V}{\partial x^2}+\left(U_1 v_1-U_2 v_2+U_1 v_2-U_1 v_2-2v_1\frac{\partial v_1}{\partial x}+\right.$$

$$\left.+2v_1\frac{\partial v_2}{\partial x}-2v_1\frac{\partial v_2}{\partial x}+2v_2\frac{\partial v_2}{\partial x}\right)\exp(-\lambda t)$$

so in (t_0, x_0)

$$0 \leqslant V(t_0, x_0) \leqslant \frac{(U_1(t_0)-U_2(t_0))v_2(t_0, x_0)\exp(-\lambda t_0)}{\lambda-U_1(t_0)+2\dfrac{\partial v_2}{\partial x}(t_0, x_0)}\,. \tag{7}$$

For $\lambda > 2\max\left|\dfrac{\partial v_2}{\partial x}\right| + \max|U_1|$ the denominator is positive, and the right side is bounded.

It is required to estimate the difference $U_1(t_0)-U_2(t_0)$. From (1.1) it follows that

$$|U_1(t)-U_2(t)|\leqslant\Big(|U_1(0)-U_2(0)|+\int\limits_0^t \varepsilon_{12}(\tau)\,d\tau\Big)\exp(\nu T),$$

where

$$\varepsilon_{12}(\tau) = \big|\, \|v_1(\tau, x)\|^2_{L^2(\Omega)}-\|v_2(\tau, x)\|^2_{L^2(\Omega)}\big|\,.$$

Such $\varepsilon_{12}(\tau)$ satisfies

$$\varepsilon_{12}(\tau) = \Big|\int\limits_{\Omega}(v_1-v_2)(v_1+v_2)\,dx\Big|\leqslant 2M\int\limits_{\Omega}|v_1-v_2)|\,dx\leqslant\pi M\varepsilon+\frac{M}{\varepsilon}\int\limits_{\Omega}(v_1-v_2)^2\,dx\,, \tag{8}$$

where $M = \max_{i=1,2}\max_{\bar{Q}}|v_i|$, $\varepsilon>0$ is arbitrary.

From Theorem 3 of [1] it follows that

$$\int\limits_{\Omega}(v_1-v_2)^2\,dx\leqslant\big(\|\varphi_1(x)-\varphi_2(x)\|^2_{L^2(\Omega)}+|U_1^2(0)-U_2^2(0)|\big)\exp(cT) \tag{9}$$

where c depends on T, ν, M. From (7), (9) and (8) with

$$\varepsilon = \max\{\|\varphi_1(x)-\varphi_2(x)\|_{L^2(\Omega)};\ \sqrt{|U_1^2(0)-U_2^2(0)|}\}$$

we obtain

$$\begin{aligned}V(t, x)\leqslant V(t_0, x_0)\leqslant\Big(\lambda-U_1(t_0)+2\frac{\partial v_2}{\partial x}(t_0, x_0)\Big)^{-1}\cdot M\exp(\nu T)\times\\ \times\big(|U_1(0)-U_2(0)|+\pi MT\max\{\|\varphi_1(x)-\varphi_2(x)\|_{L^2(\Omega)};\ \sqrt{|U_1^2(0)-U_2^2(0)|}\}+\\ +MT\big(\|\varphi_1(x)-\varphi_2(x)\|_{L^2(\Omega)}+\sqrt{|U_1^2(0)-U_2^2(0)|}\big)\exp(cT)\,,\end{aligned} \tag{10}$$

which gives the estimate for the function $W(t, x) = V(t, x)\exp(\lambda t)$.

Similarly we may estimate the least negative value of $V(t, x)$, and get

$$\begin{aligned}V(t, x)\geqslant\min\Big\{0;\ \min_{\bar{\Omega}}(\varphi_1(x)-\varphi_2(x));\ \Big(\lambda-U_1(\bar{t}_0)+2\frac{\partial v_2}{\partial x}(\bar{t}_0, \bar{x}_0)\Big)^{-1}\times\\ \times M\exp(\nu T)\cdot\Big(-|U_1(0)-U_2(0)|-\pi M\varepsilon T-\\ -\frac{MT}{\varepsilon}\big(\|\varphi_1(x)-\varphi_2(x)\|^2_{L^2(\Omega)}+|U_1^2(0)-U_2^2(0)|\big)\exp(cT)\Big)\Big\}\end{aligned}$$

and hence the proof of Lemma 2 is completed.

The following properties of solutions constructed in Theorem 2 are proved in [4];

A) From Theorem 4.1, V, 4 and Lemma 3.1, VI, 3 it follows that $\max_{\bar{Q}}|v_x|$ is bounded by a constant depending only on

$$\max|U|,\ \max_{\bar{Q}}|v|,\ \max_{\bar{\Omega}}|v_x|,\ T \text{ and } \nu\,,$$

B) From Theorem 5.4, V, 5, for any subdomain $\bar{D} \subset Q \cup Q_T$, with $d = \operatorname{dist}(D, \Gamma_T)$, the norm $\|v\|_{\bar{D}}^{2+\alpha,1+\frac{\alpha}{2}}$ is bounded by a constant depending only on d, $\max_{\bar{Q}}|v|$, $\max_{\bar{Q}}|v_x|$, $\max|U|$, $\max|U_t|$, α, T and ν.

Let us assume for a moment that the following fact is true:

C) For any function $\varphi(x) \in C^1(\bar{\Omega})$ there exists a sequence of functions $\varphi_n(x) \in C^{2+\alpha}(\bar{\Omega})$, satisfying (2), convergent to $\varphi(x)$ in $C^1(\bar{\Omega})$.

Let us consider the sequence of solutions $v_n(t, x)$ of Burgers' problem, with the initial conditions $v_n(0, x) = \varphi_n(x)$, $U(0) = U_0 =$ const. For any n this problem has a unique solution given in Theorem 2. Moreover the norms of $\varphi_n(x)$ are bounded in $C^1(\bar{\Omega})$ independently on n, so from A) and B) it follows that for any subdomain $\bar{D} \subset Q \cup \bar{Q}_T$ the norms $\|v_n\|_{\bar{D}}^{2+\alpha,1+\frac{\alpha}{2}}$ are bounded independently on n. Using the Ascoli-Arzela theorem and then the diagonal process we can find a subsequence v_{n_k} of v_n, converging in $Q \cup Q_T$ almost uniformly with the derivatives $(v_{n_k})_t$, $(v_{n_k})_x$, $(v_{n_k})_{xx}$ to a function v and its derivatives v_t, v_x, v_{xx} respectively. It is clear that the limit function v satisfies Burger's problem, moreover from A) it follows that $\max_{\bar{Q}}|v_x|$ is bounded by the same constant as $\max_{\bar{Q}}|(v_n)_x|$. From Lemma 2 it follows that v is a uniform limit of the whole sequence v_n in $C^0(\bar{Q})$, so v is continuous in $\bar{Q}$.

To complete the construction of the classical solution it remains to show that C) is satisfied.

LEMMA 3. *For any function $\varphi(x) \in C^1(\bar{\Omega})$ there exists a sequence of functions $\varphi_n(x) \in C^{2+\alpha}(\bar{\Omega})$, satisfying the conditions in Remark 2 and converging to $\varphi(x)$ in $C^1(\bar{\Omega})$.*

Proof. The condition $\varphi(0) = \varphi(\pi) = 0$ remains to extend φ on the interval $[-\pi, \pi]$ as the odd and C^1 function $\left(\frac{\varphi(x)-\varphi(0)}{x} = \frac{\varphi(-x)-\varphi(0)}{-x}\right)$. The derivative φ_x is then an even function, and will be presented as the cosine Fouriers' series

$$\varphi_x(x) = \lim_{n\to\infty} S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx\,,$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi_x(x)\cos kx\,dx\,.$$

It is known that Fejers sums

$$\sigma_n(x) = \frac{S_0(x) + \ldots + S_{n-1}(x)}{n}$$

are uniformly convergent to $\varphi_x(x)$. It is also clear that

$$\sup_{x\in[0,\pi]}\left|\int_0^x \sigma_n(y)\,dy - \varphi(x)\right| \leqslant \Pi \sup_{x\in[0,\pi]}|\sigma_n(x) - \varphi_x(x)|$$

and converges to 0 for $n \to \infty$. Now

$$\int_0^x \sigma_n(y)\,dy = \frac{1}{n}\int_0^x \left[n\frac{a_0}{2}+(n-1)a_1\cos y+\ldots+a_{n-1}\cos(n-1)y\right]dy$$

$$= \frac{1}{n}\left[n\frac{a_0}{2}x-(n-1)a_1\sin x-\ldots-\frac{a_{n-1}}{n-1}\sin(n-1)x\right]$$

and from condition $\varphi(0) = \varphi(\pi) = 0$ it follows that $\frac{a_0}{2}x = 0$, so $a_0 = 0$ and hence

$$\varphi(x) = \lim_{n\to\infty}\frac{1}{n}\left(-(n-1)a_1\sin x-\ldots-\frac{a_n-1}{n-1}\sin(n-1)x\right). \tag{11}$$

$\int_0^x \sigma_n(y)\,dy$ may be taken as the function $\varphi_n(x)$. All the conditions of Remark 2 are satisfied.

We have shown that the following theorem is satisfied:

Theorem 3. *The solution given by Theorem* 1 *is the classical solution. Moreover, if only* $\varphi(x) \in C^1(\bar{\Omega})$, *then there exists a classical solution* (U, v) *of the Burgers' problem, it is unique and continuously dependent on the initial conditions, its derivative* v_x *is bounded by the constant given in* A).

§ 3. It will be proved that the classical solution of Burgers' equation is always bounded in $L^2(\Omega)$ (the proof is also valid for the solution given in Theorem 1 of [1]).

LEMMA 4. *The solution of Burgers' equation* $U^2(t)$, $\|v(t,x)\|^2_{L^2(\Omega)}$ *is bounded for all* t.

Proof. Multiplying (1.1) by U and multiplying (1.2) by v and integrating the result over Ω, then summing together we obtain $\left(z(t) = \|v(t,x)\|^2_{L^2(\Omega)}\right)$;

$$\frac{1}{2}\frac{d}{dt}(U^2)+\frac{1}{2}\frac{d}{dt}z(t) = PU(t)-\nu U^2(t)-\nu\int_\Omega\left(\frac{\partial v}{\partial x}\right)^2 dx \leqslant PU-\nu U^2-\nu z\,, \tag{12}$$

where Lemma 1 of [1] is used;

$$\|f\|^2_{L^2(\Omega)} \leqslant \|D_x f\|^2_{L^2(\Omega)} \quad \text{for any function } f \in H_0^1(\Omega)\,.$$

Integrating (12) over $[0,T]$, for any $t \geqslant 0$ we obtain

$$\tfrac{1}{2}U^2(t)-\tfrac{1}{2}U^2(0)+\tfrac{1}{2}z(t)-\tfrac{1}{2}z(0) \leqslant P\int_0^t U(\tau)\,d\tau-\nu\int_0^t [U^2(\tau)+z(\tau)]\,d\tau$$

and denoting $L(t) = \tfrac{1}{2}U^2(t)+\tfrac{1}{2}z(t)$ we get

$$L(t) \leqslant L(0)+P\int_0^t U(\tau)\,d\tau-2\nu\int_0^t L(\tau)\,d\tau\,. \tag{13}$$

It may be noted that from (1.1) it follows that when $U(t)>\frac{P}{\nu}$ then L is decreasing, and so $\limsup_{t\to\infty} U(t)\leqslant\frac{P}{\nu}$. For all $t\geqslant 0$ we have also that $U(t)\leqslant\max\left\{U(0);\frac{P}{\nu}\right\}$.

For sufficiently large T_0, and $t\geqslant T_0$

$$U(t)\leqslant\frac{P}{\nu}+\varepsilon=:N$$

and from (13) it follows that

$$L(t)\leqslant L(0)+\int_0^t[PN-2\nu L(\tau)]\,d\tau\,. \tag{14}$$

The function L is continuous, the inequality (14) is valid also for any interval $[T_1, T_2]$, with $0\leqslant T_1\leqslant T_2$, so using the same considerations as in Remark 6 of [1], with

$$A(t)=\left[L(0)-\frac{PN}{2\nu}\right]\exp(-2\nu t)+\frac{PN}{2\nu}$$

we obtain

$$L(t)\leqslant\left[L(0)-\frac{PN}{2\nu}\right]\exp(-2\nu t)+\frac{PN}{2\nu} \tag{15}$$

which is equivalent to

$$U^2(t)+\int_\Omega v^2(t,x)\,dx\leqslant\left(U^2(0)+\int_\Omega\varphi^2(x)\,dx-\frac{PN}{\nu}\right)\exp(-2\nu t)+\frac{P^2}{\nu^2}+\frac{\varepsilon P}{\nu}$$

and hence we get the boundedness of U and z. In the same way

$$L(t)\leqslant\left[L(0)-\frac{P}{2\nu}\max\left\{U(0);\frac{P}{\nu}\right\}\right]\exp(-2\nu t)+\frac{\max\left\{U(0);\frac{P}{\nu}\right\}\cdot P}{2\nu} \tag{16}$$

and then (15), (16) imply that when $U(0)<\frac{P}{\nu}$, then $U^2(t)$ is bounded for all t by $2\left(L(0)+\frac{P^2}{2\nu^2}\right)$, and for $t\to\infty$ is "attracted" by the set $\left[0,\frac{P^2}{\nu^2}\right]$, when $U(0)\geqslant\frac{P}{\nu}$, then $U^2(t)$ is bounded by $2\left(L(0)+\frac{P}{2\nu}U(0)\right)$, and for $t\to\infty$ is "attracted" by $\left[0,\frac{P^2}{\nu^2}\right]$. The same holds true for $\|v(t,x)\|^2_{L^2(\Omega)}$.

Finally a lemma is given concerning the structure of nonstability of solutions, when $\frac{P}{\nu}>\nu$. We start with

Definition. The following function

$$K_t(f) := \frac{\|f(t,x)\|^2_{H_0^1(\Omega)}}{\|f(t,x)\|^2_{L^2(\Omega)}} \tag{17}$$

defined for the functions continuous as $[0,T] \ni t \to H_0^1(\Omega)$ with $\|f\|^2_{L^2(\Omega)} \neq 0$ is called *the complication* of f.

Shifting in (1.1) and (1.2) the function $U(t)$ on $-\frac{P}{\nu}$ we get

$$\frac{dW}{dt} = -\nu W - \int_{\Omega} v^2\,dx\,, \tag{18.1}$$

$$\frac{\partial v}{\partial t} = \left(W(t) + \frac{P}{\nu}\right) v + \nu \frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(v^2)\,. \tag{18.2}$$

Multiplying (18.1) by W, multiplying (18.2) by v and integrating over Ω, adding the results we get

$$\frac{1}{2}\left(\frac{dz}{dt} + \frac{dW^2}{dt}\right) = -\nu \|v(t,x)\|^2_{H_0^1(\Omega)} + \frac{P}{\nu} z(t) - \nu W^2(t) \tag{19}$$

now from (17), when the function $K_t(v)$ is defined

$$\frac{1}{2}\left(\frac{dz}{dt} + \frac{dW^2}{dt}\right) = -\nu K_t(v) z(t) + \frac{P}{\nu} z(t) - \nu W^2(t). \tag{20}$$

The following lemma is satisfied:

LEMMA 5, *If for some classical solution* (W, v) *a lower limit exists and satisfies*

$$\liminf_{t\to\infty} K_t(v) > \frac{P}{\nu^2}$$

then the function (W, z) *tends to* $(0,0)$ *for* $t \to \infty$.

Proof Let be $K_t(v) > \frac{P}{\nu^2} + \varepsilon$ for some $\varepsilon > 0$ and $t \geqslant T_0 \geqslant 0$. From (20) it follows that

$$\frac{1}{2}\left(\frac{dz}{dt} + \frac{dW^2}{dt}\right) \leqslant -\nu\varepsilon z(t) - \nu W^2(t) \tag{21}$$

and hence $H(t) = \frac{1}{2}(z(t) + W^2(t))$ converges to 0 for $t \to \infty$. Hence the proof is completed.

The implication of the condition given by Lemma 5 will be considered.

The solution $v(t,x)$ is given by Fouriers' series:

$$v(t,x) = \sum_{k=1}^{\infty} a_k(t) \sin kx\,,$$

$$a_k(t) = \frac{2}{\pi} \int_0^{\pi} f(t,x) \sin kx\,dx$$

and hence

$$K_t(v) = \frac{\sum_{k=1}^{\infty} k^2 a_k^2(t)}{\sum_{k=1}^{\infty} a_k^2(t)}.$$

From Lemma 5 it follows that the solutions for which the complication for big t exists and is larger than the "Reynolds' number" $\frac{P}{\nu^2}$ must tend to 0 for $t \to \infty$.

Finally we give a method of effectiv approximation of the Burgers' equation.

Remark 3. In [1] the existence of a solution was shown using the "special base", given by the own functions $V_n \in H_0^1(\Omega)$ of

$$\int_{\Omega} \frac{dV_n}{dx}\frac{dw}{dx}dx = \lambda_n \int_{\Omega} V_n w\,dx \quad \text{for any } w \in H_0^1(\Omega).$$

It is clear that the functions

$$V_n(x) = \sin nx \quad \text{with } \lambda_n = n^2$$

form such a base. Hence the proof given in Theorem 2 of [1] effectively makes it possible to approximate Burgers' equation.

The continuation of the authors' studies is prepared in "Annales Polonici Mathematici".

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