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Differentiability with respect to the parameter of solutions of operator equations of parabolic type

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Abstract. Theorems on the continuity of $\frac{\partial^s u(\lambda, t)}{\partial \lambda^s}$ of the solution $u(\lambda, t)$ of the problem $\frac{\partial u}{\partial t} + Au - \lambda u = f$, $u(\lambda, 0) = \varphi(\lambda)$ age given.

Let H be a real separable Hilbert space and let $A\colon H \supset D(A) \to H$ be a linear operator. We shall assume that $\overline{D(A)} = H$ and that A is self-adjoint, positive defined with a complete orthonormal sequence $\{u_k\}$ of eigenfunctions corresponding to the nondecreasing sequence $\{\lambda_k\}$ of eigenvalues such that $\lim_{k \to \infty} \lambda_k = \infty$.

Let

$$f\colon (R - \{\lambda_j\}) \times [0, T] \ni (\lambda, t) \to f(\lambda, t) \in H.$$

We shall consider the following problem

$$\frac{\partial u}{\partial t} + Au - \lambda u = f, \tag{1}$$

$$u(\lambda, 0) = \varphi(\lambda) \ni D(A) \quad \text{for } \lambda \in (R - \{\lambda_j\}). \tag{2}$$

By solution of (1)–(2) we mean a function

$$u\colon (R - \{\lambda_j\}) \times [0, T] \ni (\lambda, t) \to u(\lambda, t) \in D(A)$$

differentiable with respect to t in $(0, T)$ and such that (1)–(2) are satisfied. It can be proved (see e.g. [1]) that if $f(\lambda_i)$ is continuous in $[0, T]$ then the problem (1)–(2) has a unique solution.

By our assumption the solution $u(\lambda, t)$ of the problem (1)–(2) has an expansion

$$u(\lambda, t) = \sum_{n=1}^{\infty} c_n(\lambda, t) u_n, \tag{3}$$

where

$$c_n(\lambda, t) = \big(u(\lambda, t), u_n\big) \quad n = 1, 2, \ldots$$

are Fourier's coefficients of $u(\lambda, t)$ with respect to the orthonormal complete system $\{u_n\}$. We shall assume that the series (3) satisfies the assumptions of the theorem on differentation of series of functions. By this assumption we have

$$\sum_{n=1}^{\infty} \frac{\partial c_n(\lambda, t)}{\partial t} u_n + \sum_{n=1}^{\infty} c_n(\lambda, t) A u_n - \lambda \sum_{n=1}^{\infty} c_n(\lambda, t) u_n = f(\lambda, t).$$

After expanding $f(\lambda, t)$ in the Fourier series with respect to the sequence $\{u_n\}$ we obtain ordinary differential equations

(4) $$\frac{\partial c_n(\lambda, t)}{\partial t} + \lambda_n c_n(\lambda, t) - \lambda c_n(\lambda, t) = \big(f(\lambda, t), u_n\big), \quad n = 1, 2, \ldots$$

with initial conditions

(5) $$c_n(\lambda, 0) = \big(\varphi(\lambda), u_n\big), \quad n = 1, 2, \ldots$$

Clearly

(6) $$c_n(\lambda, t) = \big(\varphi(\lambda), u_n\big) e^{-(\lambda_n - \lambda)t} + \int_0^t \big(f(\lambda, \tau), u_n\big) e^{-(\lambda_n - \lambda)(t-\tau)} d\tau, \quad n = 1, 2, \ldots$$

is the solution of (4)–(5) for $n = 1, 2, \ldots$. Hence

(7) $$u(\lambda, t) = \sum_{n=1}^{\infty} \Big[\big(\varphi(\lambda), u_n\big) e^{-(\lambda_n - \lambda)t} + \int_0^t \big(f(\lambda, \tau), u_n\big) e^{-(\lambda_n - \lambda)(t-\tau)} d\tau\Big] u_n.$$

Under the same assumptions the following is obtained:

THEOREM 1. *If both* $f\colon (R - \lambda_j\}) \times [0, T] \ni (\lambda, t) \to f(\lambda, t) \in H$ *and*

$$\varphi\colon (R - \{\lambda_j\}) \ni \lambda \to \varphi(\lambda) \in H$$

are continuous and u *is the solution of* (1)–(2) *then the mapping*

$$u\colon (R - \{\lambda_j\}) \times [0, T] \ni (\lambda, t) \to u(\lambda, t) \in H$$

is continuous.

Proof. It follows from our considerations that the solution $u(\lambda, t)$ is given by (7). In view of the assumptions all terms of the Fourier series (7) are continuous with respect to (λ, t). This enables us to observe that it suffices to prove that the series (7) is uniformly convergent on every compact set contained in $(R - \{\lambda_j\}) \times [0, T]$. In order to attain this we shall prove the Cauchy condition. Set

$$B_{kl}(\lambda, t) := \sum_{n=k}^{k+l} c_n(\lambda, t) u_n,$$

where c_n, $n = 1, 2, \ldots$ are given by (6). Hence, because the sequence $\{u_n\}$ is orthonormal we get

(8) $$\|B_{k,l}(\lambda, t)\|^2 = \sum_{n=k}^{k+l} [c_n(\lambda, t)]^2.$$

On the other hand (6) implies that

$$c_n^2(\lambda, t) = \left[(\varphi(\lambda), u_n)e^{-(\lambda_n-\lambda)t} + \int_0^t (f(\lambda, \tau), u_n)e^{-(\lambda_n-\lambda)(t-\tau)}\,d\tau\right]^2 \leqslant$$

$$\leqslant 2(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t} + 2\left[\int_0^t (f(\lambda, \tau), u_n)e^{-(\lambda_n-\lambda)(t-\tau)}\,dt\right]^2$$

$$\leqslant 2(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t} + 2\int_0^t (f(\lambda, \tau), u_n)^2\,d\tau \int_0^t e^{-2(\lambda_n-\lambda)(t-\tau)}\,d\tau$$

$$= 2(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t} + 2\frac{1}{2(\lambda_n-\lambda)}(1-e^{-2(\lambda_n-\lambda)t}) \int_0^t (f(\lambda, \tau), u_n)^2\,d\tau .$$

Fix $\lambda_0 \in (R-\{\lambda_j\})$ and $\delta>0$ such that there is no eigenvalue $\lambda_j \in I .= [\lambda_0-\delta, \lambda_0+\delta]$. Let us take $k_0 \in N$ such that $\lambda_k > \lambda_0+\delta+1$. Thus we get

$$c_n^2(\lambda, t) \leqslant 2(\varphi(\lambda), u_n)^2 + \int_0^t (f(\lambda, \tau), u_n)^2\,d\tau$$

for $n>k_0$, $(\lambda, t) \in I\times[0, T]$. Hence, by the Parseval identity and by the Dini theorem series

$$\sum_{n=1}^{\infty} (\varphi(\lambda), u_n)^2, \ \sum_{n=1}^{\infty} (f(\lambda, \tau), u_n)^2$$

are uniformly convergent in I, $I\times[0, T]$, respectively. By the uniform convergence of the series $\sum\limits_{n=1}^{\infty} (f(\lambda, \tau), u_n)^2$ it follows that

$$\int_0^t \sum_{n=1}^{\infty} (f(\lambda, \tau), u_n)^2\,d\tau = \sum_{n=1}^{\infty} \int_0^t (f(\lambda, \tau), u_n)^2\,d\tau = \int_0^t \|f(\lambda, \tau)\|^2\,d\tau .$$

Hence, due to Dini's theorem, the series $\sum\limits_{n=1}^{\infty} \int\limits_0^t (f(\lambda, \tau), u_n)^2\,d\tau$ is uniformly convergent in $I\times[0, T]$. By the uniform convergence of series $\sum\limits_{n=1}^{\infty} (\varphi(\lambda), u_n)^2$ and $\sum\limits_{n=1}^{\infty} (\int\limits_0^t f(\lambda, \tau), u_n)^2\,d\tau$ it follows that

(9) $$\|B_{k,l}(\lambda, t)\| \to 0 \quad \text{when } k, l \to \infty$$

uniformly in $I\times[0, T]$. It follows from (9) that the series (7) satisfies the Cauchy condition of uniform convergence in $I\times[0, T]$. This completes the proof of Theorem 1.

THEOREM 2. *Let s be a non-negative integer. If* $\dfrac{\partial^s f(\lambda, t)}{d\lambda^s}$, $\dfrac{d^s\varphi}{\partial\lambda^s}$ *are continuous in* $(R-\{\lambda_j\}\times[0, T]$, $R-\{\lambda_j\}$, *respectively, then the solution u of the problem* (1)–(2) *has first s partial derivatives with respect to λ and* $\dfrac{\partial^s u(\lambda, t)}{\partial\lambda^s}$ *is continuous in* $(R-\{\lambda_j\}\times[0, T]$.

Proof. If $s = 0$ then Theorem 2 follows from Theorem 1. Next, in Theorem 2, we first assume $s = 1$. Assumptions of Theorem 2 imply that all terms of the series (7) are of class C^1 with respect to λ. Then we can write

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial \lambda}[c_n(\lambda, t)u_n)] = \sum_{n=1}^{\infty} \Big\{\Big(\frac{d\varphi(\lambda)}{d\lambda}, u_n\Big)e^{-(\lambda_n-\lambda)t} + (\varphi(\lambda), u_n)e^{-(\lambda_n-\lambda)t}t + \int_0^t \Big[\Big(\frac{\partial f(\lambda, \tau)}{\partial \lambda}, u_n\Big)e^{-(\lambda_n-\lambda)(t-\tau)} + (f(\lambda, \tau), u_n)e^{-(\lambda_n-\lambda)(t-\tau)}(t-\tau)\Big]d\tau\Big\}u_n. \tag{10}$$

We shall show that the series (10) is uniformly convergent in the set $I\times[0; T]$ by proving the Cauchy condition. Write

$$D_{k,l}(\lambda, t) := \sum_{n=k}^{k+l} \frac{\partial c_n(\lambda, t)}{\partial \lambda}u_n,$$

where c_n was defined in (6). Since the sequence $\{u_n\}$ is orthonormal in H it follows that

$$\|D_{k,l}(\lambda, t)\|^2 = \sum_{n=k}^{k+1} \Big[\frac{\partial c_n(\lambda, t)}{\partial \lambda}\Big]^2. \tag{11}$$

We infer that

$$\begin{aligned}
\Big[\frac{c_n(\lambda, t)}{\partial \lambda}\Big]^2 &= \Big\{\Big(\frac{d\varphi(\lambda)}{d\lambda}, u_n\Big)e^{-(\lambda_n-\lambda)t} + (\varphi(\lambda), u_n)e^{-(\lambda_n-\lambda)t}t + \\
&\quad + \int_0^t \Big[\Big(\frac{\partial f(\lambda, \tau)}{\partial \lambda}, u_n\Big)e^{-(\lambda_n-\lambda)(t-\tau)} + (f(\lambda, \tau), u_n)e^{-(\lambda_n-\lambda)(t-\tau)}(t-\tau)\Big]d\tau\Big\}^2 \\
&\leqslant 4\Big(\frac{d\varphi(\lambda)}{\partial \lambda}, u_n\Big)^2 e^{-2(\lambda_n-\lambda)t} + 4(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t}t^2 + \\
&\quad + 4\Big[\int_0^t \Big(\frac{\partial f(\lambda, \tau)}{\partial \lambda}, u_n\Big)e^{-(\lambda_n-\lambda)(t-\tau)}d\tau\Big]^2 + 4\Big[\int_0^t (f(\lambda, \tau), u_n)e^{-(\lambda_n-\lambda)(t-\tau)}(t-\tau)d\tau\Big]^2 \\
&\leqslant 4\Big(\frac{\partial\varphi(\lambda)}{\partial \lambda}\ u_n\Big)^2 e^{-2(\lambda_n-\lambda)t} + 4(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t}t^2 + \\
&\quad + 4\int_0^t \Big(\frac{\partial f(\lambda, \tau)}{\partial \lambda}, u_n\Big)^2 d\tau \int_0^t e^{-2(\lambda_n-\lambda)(t-\tau)}d\tau + \\
&\quad + 4\int_0^t (f(\lambda, \tau), u_n)^2 d\tau \int_0^t e^{-2(\lambda_n-\lambda)(t-\tau)}(t-\tau)^2 d\tau
\end{aligned}$$

$$\leqslant 4\left(\frac{\partial\varphi(\lambda)}{\partial\lambda}, u_n\right)^2 e^{-2(\lambda_n-\lambda)t} + 4(\varphi(\lambda), u_n)^2 e^{-2(\lambda_n-\lambda)t} t^2 +$$

$$+4\frac{1}{2(\lambda_n-\lambda)}(1-e^{-2(\lambda_n-\lambda)t})\int_0^t\left(\frac{\partial f(\lambda,\tau)}{\partial\lambda}, u_n\right)^2 d\tau +$$

$$+16T^2\frac{1}{2(\lambda_n-\lambda)}\left(1-e^{-2(\lambda_n-\lambda)t}\right)\int_0^t (f(\lambda,\tau), u_n)^2 d\tau\,.$$

Next, as in the proof of Theorem 1, choose I and k_0 such that $\lambda k_0 > \lambda_0 + \delta + 1$. Then for $n > k_0$, $\lambda \in I$ we get

$$\left[\frac{\partial c_n(\lambda, t)}{\partial\lambda}\right]^2 \leqslant 4\left(\frac{\partial\varphi(\lambda)}{\partial\lambda}, u_n^2\right) + 4(\varphi(\lambda), u_n)^2 + 2\int_0^t\left(\frac{\partial f(\lambda,\tau)}{\partial\lambda}, u_n\right)^2 d\tau +$$

$$+8T^2\int_0^t (f(\lambda,\tau), u_n)^2 d\tau \quad \text{for } (\lambda, t) \in I\times[0, T]\,.$$

By Parseval's identity and Dini's theorem we therefore obtain that the following series

$$\sum_{n=1}^{\infty}(\varphi(\lambda), u_n)^2, \ \sum_{n=1}^{\infty}\left(\frac{\partial\varphi(\lambda)}{\partial\lambda}, u_n\right)^2, \ \sum_{n=1}^{\infty}(f(\lambda,\tau), u_n)^2, \ \sum_{n=1}^{\infty}\left(\frac{\partial f(\lambda,\tau)}{\partial\lambda}, u_n\right)^2$$

are uniformly convergent in I, I, $I\times[0, T]$, $I\times[0, T]$, respectively. Now, the theorem on integration of series and Dini's theorem imply that series $\sum_{n=1}^{\infty}\int_0^t (f(\lambda,\tau), u_n)^2 d\tau$ and $\sum_{n=1}^{\infty}\int_0^t\left(\frac{\partial f(\lambda,\tau)}{\partial\lambda}, u_n\right)^2 d\tau$ are uniformly convergent in $I\times[0, T]$ and we conclude from this that

$$\|D_{k,l}(\lambda, t)\|^2 \to 0, \quad \text{when } k, l\to\infty \tag{12}$$

uniformly in $I\times[0, T]$. Therefore, the series (12) is uniformly convergent in $I\times[0, T]$. Hence, by the theorem of differentiation of functional series

$$\frac{\partial u(\lambda, t)}{\partial\lambda} = \frac{\partial}{\partial\lambda}\sum_{n=1}^{\infty} c_n(\lambda, t) u_n = \sum_{n=1}^{\infty}\frac{\partial c_n(\lambda, t)}{\partial\lambda} u_n \tag{13}$$

and $\frac{\partial u(\lambda, t)}{\partial\lambda}$ is continuous in its domain of definition. Now the conclusion of Theorem 2 can be obtained by induction with respect to s and by the same argument as in the case of $s = 1$.

Theorems 1, 2 were formulated for a general linear equation of the form (1) in a Hilbert space H. But a suitable change of the operator A permit us to apply these theorems to many typical problems of mathematical physics. For example, the boundary problem for a strong parabolic system of equations will be presented.

Let $G\subset R^n$ be a bounded domain with a sufficiently regular boundary ∂G. Suppose that $L\left(x, \frac{\partial}{\partial x}\right)$ is a square $N\times N$ matrix in which elements L_{ij} are differential operators

$$L_{ij}\left(x, \frac{\partial}{\partial x}\right) = \sum_{|\alpha|\leqslant 2m} a_{ij}^{(\alpha)}(x)\, D^\alpha \tag{14}$$

where $\alpha = (\alpha_1, \alpha_2, ., \alpha_n)$ $|\alpha| = \alpha_1+\ldots+\alpha_n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}$. Assume that $a_{ij}^{(\alpha)}$ are real sufficiently regular functions in the domain G and that

$$(-1)^m \sum_{i,j=1}^{N} \sum_{|\alpha|=2m} a_{ij}^{(\alpha)}(x)\zeta_1^{\alpha_1}\ldots\zeta_n^{\alpha_n}\eta_i\eta_j>0 \tag{15}$$

for all $x\in G$, $\zeta_k\in R$, $\eta_i\in R$, $k, i = 1, 2, \ldots, n$ such that

$$\sum_{k=1}^{n} \zeta_k^2 \neq 0 \quad \text{and} \quad \sum_{k=1}^{n} \eta_i^2 \neq 0 .$$

Let H be the Hilbert space of all vector functions $u = (u_1, \ldots, u_n)$ such that $u_j\in L^2(G)$, for $j = 1, \ldots, n$. Let $D(L)$ be the subspace of H of all functions u of class C^{2m} in $\bar{G}$ such that

$$u\Big|_{\partial G} = \frac{\partial u}{\partial n}\Big|_{\partial G} = \ldots = \frac{\partial^{m-1}}{\partial n^{m-1}}\Big|_{\partial G} = 0 , \tag{16}$$

where n is the normal direction to ∂G. Let L: $D(L)\to H$ be the linear operator defined by the matrix L and suppose that L is symmetric. It may be proved that L can be extended to a self-adjoint operator in H. Denote this extension by A. In this case theorems 1, 2 are theorems on continuity and differentiability with respect to the parameter of solution of a strong parabolic system of equations of the order $2m$.

References

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