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A note on semi-groups of the C_ϱ class

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Let H be a complex Hilbert space with inner product (x, v): $x, v \in H$ and norm $\|x\| = \sqrt{(x, x)}$: $x \in H$. $L(H)$ denotes the algebra of all linear bounded operators on H. $T \in L(H)$ belongs to the C_ϱ class iff there is a Hilbert space K including H and a unitary operator $U \in L(K)$ such that $T^n x = PU^n x$ for $x \in H$ and $n = 1, 2, \ldots$, where P denotes the orthogonal projection of K onto H. If $\varrho = 1$ then T is a contraction, i.e. $\|T\| \leqslant 1$ and if T is a contraction then T is in the C_1 class (see [1]). It is proved in [1], that T is in the C_ϱ class if and only if

$$(\varrho-2)\, \|(I-zT)x\|^2 + 2\operatorname{Re}((I-zT)x, x) \geqslant 0$$

for every $x \in H$ and $|z| \leqslant 1$. It follows that if T is in the C_ϱ class then for every $\lambda > \varrho$, T is in the C_λ class.

Let $T(t)$ be a strongly continuous one-parameter semi-group of operators on the space H, i.e. T: $R^+ \to L(H)$ such that

$$T(t+s) = T(t)T(s) \quad \text{for every } t, s \geqslant 0$$

$$T(t)x \to x \quad \text{for } t \to 0 \quad \text{and every } x \in H.$$

Define

$$\varrho(t) = \begin{cases} \infty \text{ if } T(t) \text{ is not in the } C_\varrho \text{ class for every } \varrho > 0 \\ \inf\{\varrho\colon T(t) \text{ is in the } C_\varrho \text{ class}\}. \end{cases}$$

Then we have

THEOREM. *The following conditions are equivalent*:
1. *there is a constant* $M > 0$ *such that* $\varrho(t) < M$ *for every* $t > 0$,
2. *there are* $t_0 > 0$ *and* $M > 0$ *such that* $\varrho(t) < M$ *for every* $t \in (0, t_0)$,
3. *there is a sequence* t_n *such that* $t_n \to 0$, $t_n > 0$ *and* $\varrho(t_n)$ *is bounded,*
4. $\varrho(t) \leqslant 1$ *for every* $t > 0$ *i.e.* $T(t)$ *is a semi-group of contractions.*

Proof. Because implications $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$ are evident we have to prove only the implication $3 \Rightarrow 4$. Let $\varrho(t_n) < M$ for every $n (M > 0)$. By the definition of $\varrho(t)$ we know that $T(t_n)$ is in the $C_{\varrho(t_n)}$ class for every n. It follows that for every $x \in H$ the following inequality holds true:

$$(\varrho(t_n)-2)\, \|(I-T(t_n))x\|^2 + 2\operatorname{Re}((I-T(t_n))x, x) \geqslant 0.$$

Consequently we get

$$2\,\mathrm{Re}\big((T(t_n)-I)x,\, x\big) \leqslant \big(\varrho(t_n)-2\big)\,\|(T(t_n)-I)x\|^2 \leqslant (M-2)\big((T(t_n)-I)x,\, (T(t_n)-I)x\big)\,.$$

Since $t_n > 0$ we have

$$2\,\mathrm{Re}\left(\frac{1}{t_n}(T(t_n)-I)x,\, x\right) \leqslant (M-2)\left(\frac{1}{t_n}(T(t_n)-I)x,\, (T(t_n)-I)x\right).$$

If $x \in D(A)$, where A is the generator of the semi-group $T(t)$, by the definition of A and continuity of $T(t)$ we get $2\,\mathrm{Re}(Ax, x) \leqslant (M-2)(Ax, 0) = 0$. Because the condition $\mathrm{Re}(Ax, x) \leqslant 0$ for every $x \in D(A)$ implies that $T(t)$ is a semi-group of contractions (see [1]) our proof is complete.

References

[1] B. Sz.-Nagy, C. Foias, *Analyse harmonique des operateurs de l'espace de Hilbert*, Masson et C^ie^ Academiai Kiado, (1967).

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