

Unitary dilation of two-parameter semi-groups of contractions II

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In what follows, H is a complex Hilbert space with inner product (x, y) ; $x, y \in H$ and norm $\|x\| = \sqrt{(x, x)}$; $x \in H$. $L(H)$ denotes the algebra of all linear bounded operators on H . Let G be a semi-group. The map $T: G \rightarrow L(H)$ is called a semi-group of operators iff $T(t+s) = T(t)T(s)$ for every $t, s \in G$. The semi-group is called strongly continuous iff $T(s)x \rightarrow x$ for $s \rightarrow 0$ and for every $x \in H$. The semi-group is called one-parameter if $G = R^+$ and two-parameter if $G = \{(t, s): t, s \geq 0\}$. We have proved in [3] that every two-parameter strongly continuous semi-group of contractions has a unitary dilation, i.e. there is a space K including H and unitary strongly continuous semi-group $U(t, s)$ on K such that $T(t, s)x = PU(t, s)x$ for $t, s \geq 0$ and $x \in H$, where P is the projection of K onto H . In this paper we give a second proof of this fact. The proof is geometrical and we do not use the theorem of the spectral dilation of a semi-spectral measure. We use the Parrott construction of a unitary dilation of a pair of contractions.

The Parrott construction is the following:

Suppose T_1 and T_2 are commuting contractions on H and the pair U_1, U_2 on K is their unitary dilation. Let $K_0 = \bigvee_{n=-\infty}^{\infty} U_2^n H$. Evidently K_0 reduces U_2 . Define operators S_1 and S_2 as follows:

$$S_2 = U_2|_{K_0} \quad \text{and} \quad S_1 = P_{K_0} U_1|_{K_0}.$$

S_2 is the minimal unitary dilation of T_2 . Parrott has proved that

$$S_1 S_2 = S_2 S_1$$

$T_1^n T_2^m x = P_H S_1^n S_2^m x$ for every $x \in H$ and $n, m = 0, 1, 2, \dots$, $S_1 K_0^+ \subset K_0^+$ and $S_1(K_0^+ \ominus H) \subset K_0^+ \ominus H$, where $K_0^+ = \bigvee_{n=0}^{\infty} S_2^n H$.

$$\|S_1\| = \|T_1\|.$$

If V_1 is the minimal unitary dilation of S_1 on K_1 then there is exactly one unitary extension V_2 on K_1 of S_2 such that $V_1 V_2 = V_2 V_1$ and K_0 reduces V_2 . Since for $x \in H$ and $n, m = 0, 1, 2, \dots$ we have

$$P_H V_1^n V_2^m x = P_H P_{K_0} V_1^n V_2^m x = P_H P_{K_0} V_1^n S_2^m x = P_H S_1^n S_2^m x = T_1^n T_2^m x$$

hence the pair V_1, V_2 is a unitary dilation of T_1, T_2 .

We start from the following lemma:

LEMMA 1. Let U be a unitary operator on $K = \bigvee_{n=-\infty}^{\infty} U^n H$, and let S on K commute with U . Then if $\lambda \in \sigma_p(S)$ there is $k_1 \in K$ such that $P_H k_1 \neq 0$ and $Sk_1 = \lambda k_1$.

Proof. Since $\lambda \in \sigma_p(S)$, there is $k \in K, k \neq 0$ such that $Sk = \lambda k$. The unitarity of U implies that U and S doubly commute, i.e. S commutes with U and U^* . It follows that for every integer number n we have

$$(1) \quad SU^n k = U^n Sk = \lambda U^n k.$$

Let $K^+ = \bigvee_{n=0}^{\infty} U^n H$. We consider two situations:

- I. $P_{K^+} k \neq 0$
- II $P_{K^+} k = 0$

In the first case the set $N = \{n \in \mathbb{Z}: n \geq 0 \text{ and } P_{U^n H} k \neq 0\}$ is not empty because $P_{K^+} k \neq 0$ and is, by definition, bounded below by zero. Define $n_0 = \min N$. Now let $k_1 = U^{-n_0} k$. It follows by (1) that $Sk_1 = \lambda k_1$. By the definition of n_0 we conclude that $P_{U^{n_0} H} k \neq 0$. Consequently there is $h \in H$ such that $(k, U^{n_0} h) \neq 0$. It follows that $(k_1, h) = (U^{-n_0} k, h) = (k, U^{n_0} h) \neq 0$ hence $P_H k_1 \neq 0$.

In the second case, the set $M = \{m \in \mathbb{Z}: P_{U^m H} k \neq 0\}$ is not empty and M is bounded above by zero. Let $m_0 = \max M$ and $k_1 = U^{-m_0} k$. By (1) we have $Sk_1 = \lambda k_1$. The same computation as in case I shows that $P_H k_1 \neq 0$ which finishes the proof of our lemma.

Using this lemma we shall prove the following proposition:

PROPOSITION 1. Let T_1 and T_2 be a pair of commuting contractions on H and let the pair U_1, U_2 be a unitary dilation of T_1 and T_2 . Suppose that $\lambda \notin \sigma_p(T_1)$ and $|\lambda| = 1$. If $S_1 = P_{K_0} U_1|_{K_0}$, where $K_0 = \bigvee_{n=-\infty}^{\infty} U_2^n H$ then $\lambda \notin \sigma_p(S_1)$.

Proof. Suppose that there is $k \in K, k \neq 0$ such that $S_1 k = \lambda k$. It follows by Lemma 1 that there is $k_1 \in K_0$ such that $P_H k_1 \neq 0$ and $S_1 k_1 = \lambda k_1$. Let $h = P_H k_1, f = P_{K_0^+ \ominus H} k_1$ and $g = P_{K_0 \ominus K_0^+} k_1$. Then $k_1 = h + f + g$ and $\lambda(h + f + g) = S_1(h + f + g)$. Since $h + f \in K_0^+$ and $S_1 K_0^+ \subset K_0^+$ (see the Parrott construction) we have $\lambda(h + f) = S_1(h + f) + P_{K_0^+} S_1 g$ and $\lambda g = P_{K_0 \ominus K_0^+} S_1 g$. We shall prove that $P_{K_0^+} S_1 g = 0$. Suppose that $\|P_{K_0^+} S_1 g\| > 0$. It follows $\|S_1 g\|^2 = \|P_{K_0^+} S_1 g\|^2 + \|P_{K_0 \ominus K_0^+} S_1 g\|^2 > \|P_{K_0 \ominus K_0^+} S_1 g\|^2 = \|\lambda g\|^2 = \|g\|^2$. Consequently $\|g\| < \|S_1 g\|$. Since $\|S_1\| = \|T_1\| \leq 1$, hence $\|g\| \geq \|S_1 g\|$ and we have a contradiction. We have proved that $\lambda(h + f) = S_1(h + f)$. Since $f \in K_0^+ \ominus H$ and $S_1(K_0^+ \ominus H) \subset K_0^+ \ominus H$, hence $\lambda h = P_H S_1 h$. But the pair S_1, S_2 is a dilation of T_1, T_2 and in particular we have $\lambda h = P_H S_1 h = T_1 h$. But $h = P_H k_1$ is a nonzero vector. It follows that $\lambda \in \sigma_p(T_1)$ which contradicts the assumption of our proposition. The proof is complete.

We shall also prove the following:

LEMMA 2. Suppose that T_1 and T_2 are commuting contractions on H . Let S_1, S_2 on K_0 and V_1, V_2 on K_1 be as in the Parrott construction. If $\lambda \notin \sigma_p(S_2)$ then $\lambda \notin \sigma_p(V_2)$.

