

## Unitary dilation of two-parameter semi-groups of contractions II

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In what follows,  $H$  is a complex Hilbert space with inner product  $(x, y)$ ;  $x, y \in H$  and norm  $\|x\| = \sqrt{(x, x)}$ ;  $x \in H$ .  $L(H)$  denotes the algebra of all linear bounded operators on  $H$ . Let  $G$  be a semi-group. The map  $T: G \rightarrow L(H)$  is called a semi-group of operators iff  $T(t+s) = T(t)T(s)$  for every  $t, s \in G$ . The semi-group is called strongly continuous iff  $T(s)x \rightarrow x$  for  $s \rightarrow 0$  and for every  $x \in H$ . The semi-group is called one-parameter if  $G = R^+$  and two-parameter if  $G = \{(t, s): t, s \geq 0\}$ . We have proved in [3] that every two-parameter strongly continuous semi-group of contractions has a unitary dilation, i.e. there is a space  $K$  including  $H$  and unitary strongly continuous semi-group  $U(t, s)$  on  $K$  such that  $T(t, s)x = PU(t, s)x$  for  $t, s \geq 0$  and  $x \in H$ , where  $P$  is the projection of  $K$  onto  $H$ . In this paper we give a second proof of this fact. The proof is geometrical and we do not use the theorem of the spectral dilation of a semi-spectral measure. We use the Parrott construction of a unitary dilation of a pair of contractions.

The Parrott construction is the following:

Suppose  $T_1$  and  $T_2$  are commuting contractions on  $H$  and the pair  $U_1, U_2$  on  $K$  is their unitary dilation. Let  $K_0 = \bigvee_{n=-\infty}^{\infty} U_2^n H$ . Evidently  $K_0$  reduces  $U_2$ . Define operators  $S_1$  and  $S_2$  as follows:

$$S_2 = U_2|_{K_0} \quad \text{and} \quad S_1 = P_{K_0} U_1|_{K_0}.$$

$S_2$  is the minimal unitary dilation of  $T_2$ . Parrott has proved that

$$S_1 S_2 = S_2 S_1$$

$T_1^n T_2^m x = P_H S_1^n S_2^m x$  for every  $x \in H$  and  $n, m = 0, 1, 2, \dots$ ,  $S_1 K_0^+ \subset K_0^+$  and  $S_1(K_0^+ \ominus H) \subset K_0^+ \ominus H$ , where  $K_0^+ = \bigvee_{n=0}^{\infty} S_2^n H$ .

$$\|S_1\| = \|T_1\|.$$

If  $V_1$  is the minimal unitary dilation of  $S_1$  on  $K_1$  then there is exactly one unitary extension  $V_2$  on  $K_1$  of  $S_2$  such that  $V_1 V_2 = V_2 V_1$  and  $K_0$  reduces  $V_2$ . Since for  $x \in H$  and  $n, m = 0, 1, 2, \dots$  we have

$$P_H V_1^n V_2^m x = P_H P_{K_0} V_1^n V_2^m x = P_H P_{K_0} V_1^n S_2^m x = P_H S_1^n S_2^m x = T_1^n T_2^m x$$

hence the pair  $V_1, V_2$  is a unitary dilation of  $T_1, T_2$ .

We start from the following lemma:

LEMMA 1. Let  $U$  be a unitary operator on  $K = \bigvee_{n=-\infty}^{\infty} U^n H$ , and let  $S$  on  $K$  commute with  $U$ . Then if  $\lambda \in \sigma_p(S)$  there is  $k_1 \in K$  such that  $P_H k_1 \neq 0$  and  $Sk_1 = \lambda k_1$ .

Proof. Since  $\lambda \in \sigma_p(S)$ , there is  $k \in K, k \neq 0$  such that  $Sk = \lambda k$ . The unitarity of  $U$  implies that  $U$  and  $S$  doubly commute, i.e.  $S$  commutes with  $U$  and  $U^*$ . It follows that for every integer number  $n$  we have

$$(1) \quad SU^n k = U^n Sk = \lambda U^n k.$$

Let  $K^+ = \bigvee_{n=0}^{\infty} U^n H$ . We consider two situations:

- I.  $P_{K^+} k \neq 0$
- II  $P_{K^+} k = 0$

In the first case the set  $N = \{n \in \mathbb{Z}: n \geq 0 \text{ and } P_{U^n H} k \neq 0\}$  is not empty because  $P_{K^+} k \neq 0$  and is, by definition, bounded below by zero. Define  $n_0 = \min N$ . Now let  $k_1 = U^{-n_0} k$ . It follows by (1) that  $Sk_1 = \lambda k_1$ . By the definition of  $n_0$  we conclude that  $P_{U^{n_0} H} k \neq 0$ . Consequently there is  $h \in H$  such that  $(k, U^{n_0} h) \neq 0$ . It follows that  $(k_1, h) = (U^{-n_0} k, h) = (k, U^{n_0} h) \neq 0$  hence  $P_H k_1 \neq 0$ .

In the second case, the set  $M = \{m \in \mathbb{Z}: P_{U^m H} k \neq 0\}$  is not empty and  $M$  is bounded above by zero. Let  $m_0 = \max M$  and  $k_1 = U^{-m_0} k$ . By (1) we have  $Sk_1 = \lambda k_1$ . The same computation as in case I shows that  $P_H k_1 \neq 0$  which finishes the proof of our lemma.

Using this lemma we shall prove the following proposition:

PROPOSITION 1. Let  $T_1$  and  $T_2$  be a pair of commuting contractions on  $H$  and let the pair  $U_1, U_2$  be a unitary dilation of  $T_1$  and  $T_2$ . Suppose that  $\lambda \notin \sigma_p(T_1)$  and  $|\lambda| = 1$ . If  $S_1 = P_{K_0} U_1|_{K_0}$ , where  $K_0 = \bigvee_{n=-\infty}^{\infty} U_2^n H$  then  $\lambda \notin \sigma_p(S_1)$ .

Proof. Suppose that there is  $k \in K, k \neq 0$  such that  $S_1 k = \lambda k$ . It follows by Lemma 1 that there is  $k_1 \in K_0$  such that  $P_H k_1 \neq 0$  and  $S_1 k_1 = \lambda k_1$ . Let  $h = P_H k_1, f = P_{K_0^+ \ominus H} k_1$  and  $g = P_{K_0 \ominus K_0^+} k_1$ . Then  $k_1 = h + f + g$  and  $\lambda(h + f + g) = S_1(h + f + g)$ . Since  $h + f \in K_0^+$  and  $S_1 K_0^+ \subset K_0^+$  (see the Parrott construction) we have  $\lambda(h + f) = S_1(h + f) + P_{K_0^+} S_1 g$  and  $\lambda g = P_{K_0 \ominus K_0^+} S_1 g$ . We shall prove that  $P_{K_0^+} S_1 g = 0$ . Suppose that  $\|P_{K_0^+} S_1 g\| > 0$ . It follows  $\|S_1 g\|^2 = \|P_{K_0^+} S_1 g\|^2 + \|P_{K_0 \ominus K_0^+} S_1 g\|^2 > \|P_{K_0 \ominus K_0^+} S_1 g\|^2 = \|\lambda g\|^2 = \|g\|^2$ . Consequently  $\|g\| < \|S_1 g\|$ . Since  $\|S_1\| = \|T_1\| \leq 1$ , hence  $\|g\| \geq \|S_1 g\|$  and we have a contradiction. We have proved that  $\lambda(h + f) = S_1(h + f)$ . Since  $f \in K_0^+ \ominus H$  and  $S_1(K_0^+ \ominus H) \subset K_0^+ \ominus H$ , hence  $\lambda h = P_H S_1 h$ . But the pair  $S_1, S_2$  is a dilation of  $T_1, T_2$  and in particular we have  $\lambda h = P_H S_1 h = T_1 h$ . But  $h = P_H k_1$  is a nonzero vector. It follows that  $\lambda \in \sigma_p(T_1)$  which contradicts the assumption of our proposition. The proof is complete.

We shall also prove the following:

LEMMA 2. Suppose that  $T_1$  and  $T_2$  are commuting contractions on  $H$ . Let  $S_1, S_2$  on  $K_0$  and  $V_1, V_2$  on  $K_1$  be as in the Parrott construction. If  $\lambda \notin \sigma_p(S_2)$  then  $\lambda \notin \sigma_p(V_2)$ .

**Proof.** Suppose that there is  $k \in K_1$  such that  $k \neq 0$  and  $V_1 k = \lambda k$ . It follows by Lemma 1 that there is  $k_1 \in K_1$  such that  $P_{K_0} k_1 \neq 0$  and  $V_2 k_1 = \lambda k_1$ . Let  $h = P_{K_0} k_1$ . Since  $K_0$  reduces  $V_2$  we have  $\lambda h = V_2 h = S_2 h$  which contradicts the assumption that  $\lambda \notin \sigma_p(S_2)$ .

Since for a single contraction  $T$ ;  $\sigma_p(T) \cap \Gamma = \sigma_p(U)$  where  $U$  is the minimal unitary dilation of  $T$  (see Theorem II-6-1 of [1]) by Lemma 2 and Proposition 1 we have

**THEOREM 1.** *Suppose that  $T_1$  and  $T_2$  are commuting contractions on  $H$ . Let  $V_1$  and  $V_2$  on  $K_1$  be as in the Parrott construction. Then*

$$\sigma_p(T_i) \cap \Gamma = \sigma_p(V_i) \quad (i = 1, 2)$$

**Proof.** If  $\lambda_i \in \sigma_p(T_i) \cap \Gamma$  then there is  $h_i \in H$ ,  $h_i \neq 0$  such that  $T_i h_i = \lambda_i h_i$ . Now we have

$$\begin{aligned} \|h_i\|^2 &= \|V_i h_i\|^2 = \|P_H V_i h_i\|^2 + \|P_{K_1 \ominus H} V_i h_i\|^2 = \|T_i h_i\|^2 + \|P_{K_1 \ominus H} V_i h_i\|^2 \\ &= \|\lambda_i h_i\|^2 + \|P_{K_1 \ominus H} V_i h_i\|^2 = \|h_i\|^2 + \|P_{K_1 \ominus H} V_i h_i\|^2. \end{aligned}$$

It follows  $P_{K_1 \ominus H} V_i h_i = 0$  and consequently  $V_i h_i = P_H V_i h_i = T_i h_i = \lambda_i h_i$ . We have proved that if  $\lambda_i \in \sigma_p(T_i) \cap \Gamma$  then  $\lambda_i \in \sigma_p(V_i)$ . Now we shall prove that if  $\lambda_i \notin \sigma_p(T_i)$  and  $|\lambda_i| = 1$  then  $\lambda_i \notin \sigma_p(V_i)$ , which will complete the proof. Since  $\lambda_2 \notin \sigma_p(T_2)$  and  $|\lambda_2| = 1$  and  $S_2$  is the minimal unitary dilation of  $T_2$  hence  $\lambda_2 \notin \sigma_p(S_2)$ . Now by Lemma 2 we get  $\lambda_2 \notin \sigma_p(V_2)$ . By Proposition 1 we have that  $\lambda_1 \notin \sigma_p(S_1)$  because  $\lambda_1 \notin \sigma_p(T_1)$  and  $|\lambda_1| = 1$ . Since  $V_1$  is the minimal unitary dilation of  $S_1$  hence  $\lambda_1 \notin \sigma_p(V_1)$ , which finishes the proof.

Now we can prove our main theorem,

**THEOREM 2.** *Let  $T(t, s)$  be a two-parameter strongly continuous semi-group of contractions on  $H$ . Suppose that  $T_i$  is the cogenerator of  $T_i(t)$ , where  $T_1(t) = T(t, 0)$  and  $T_2(s) = T(0, s)$ . Let the pair  $V_1, V_2$  be a unitary dilation of  $T_1$  and  $T_2$  as in the Parrott construction. Then*

$$(i) \quad 1 \notin \sigma_p(V_i) \quad (i = 1, 2)$$

$$(ii) \quad V(t, s) = V_1(t) V_2(s) \text{ is the unitary dilation of } T(t, s), \text{ where } V_i(t) = e_i(V_i)$$

$$(i = 1, 2), \quad e_i(z) = \exp\left(t \frac{z+1}{z-1}\right)$$

$$(iii) \quad K_1 = \bigvee_{n=-\infty}^{\infty} V_1^n K_0 = \bigvee_{s=-\infty}^{\infty} V_1(s) K_0$$

$$(iv) \quad K_0 = \bigvee_{n=-\infty}^{\infty} V_2^n H = \bigvee_{s=-\infty}^{\infty} V_2(s) H.$$

**Proof.** Since  $T_1$  and  $T_2$  are cogenerators,  $1 \notin \sigma_p(T_i)$  (see [1]) ( $i = 1, 2$ ). It follows by Theorem 1 that  $1 \notin \sigma_p(V_i)$ . Consequently (see [1])  $V_i(t) = e_i(V_i)$  is a unitary one-parameter strongly continuous semi-group. Hence  $V(t, s) = V_1(t) V_2(s)$  is a unitary strongly continuous two-parameter semi-group. Now we have to prove that  $V(t, s)$  is a dilation of  $T(t, s)$ . Since the pair  $V_1, V_2$  is a unitary dilation of  $T_1, T_2$ , for arbitrary polynomials  $p, q$  and every  $x \in H$  we have the equality

$$p(T_1)q(T_2)x = P_H p(V_1)q(V_2)x.$$

By limit passage we may conclude that for every  $f, g \in A(D)$  and every  $x \in H$

$$f(T_1)g(T_2)x = P_H f(V_1)g(V_2)x.$$

In particular for every  $x \in H$  and  $0 < r < 1$  we have

$$(2) \quad e_t(rT_1)e_s(rT_2)x = P_H e_t(rV_1)e_s(rV_2)x.$$

Since  $1 \notin \sigma_p(T_i)$  and  $1 \notin \sigma_p(V_i)$  hence  $e_t(rT_i)x \rightarrow e_t(T_i)x$  and  $e_t(rV_i)x \rightarrow e_t(V_i)x$  for  $r \rightarrow 1$  and  $x \in H$ . An easy computation (see [3]) shows that

$$e_t(rT_1)e_s(rT_2)x \rightarrow e_t(T_1)e_s(T_2)x \quad \text{and} \quad e_t(rV_1)e_s(rV_2)x \rightarrow e_t(V_1)e_s(V_2)x$$

for  $r \rightarrow 1$  and  $x \in H$ .

It follows by limit passage in (2) that

$$T(t, s) = e_t(T_1)e_s(T_2)x = P_H e_t(V_1)e_s(V_2)x = P_H V(t, s)x$$

for every  $x \in H$  and  $t, s \geq 0$ , hence  $V(t, s)$  is a unitary dilation of  $T(t, s)$ . We shall show that the equalities (iii) and (iv) hold true. Since  $V_1$  is the minimal unitary dilation of  $S_1$ ,  $S_1$  and  $S_2$  as in the Parrott construction, for the one-parameter semi-group  $V_1(t)$  we have (see [1])

$$K_1 = \bigvee_{n=-\infty}^{\infty} V_1^n K_0 = \bigvee_{s=-\infty}^{\infty} V_1(s)K_0.$$

Since  $K_0$  reduces  $V_2$  and  $V_2|_{K_0} = S_2$ , then the semi-group  $S_2(s) = e_s(S_2)$  is equal to  $V_2(s)|_{K_0}$ . For the semi-group  $S_2(s)$  we have

$$K_0 = \bigvee_{n=-\infty}^{\infty} S_2^n H = \bigvee_{s=-\infty}^{\infty} S_2(s)H$$

because  $S_2$  is the minimal unitary dilation of  $T_2$ . It follows

$$\bigvee_{n=-\infty}^{\infty} S_2^n H = \bigvee_{n=-\infty}^{\infty} (V_2|_{K_0})^n H = \bigvee_{s=-\infty}^{\infty} V_2^n H$$

and analogically

$$\bigvee_{s=-\infty}^{\infty} S_2(s)H = \bigvee_{s=-\infty}^{\infty} V_2(s)|_{K_0} H = \bigvee_{s=-\infty}^{\infty} V_2(s)H.$$

Consequently  $K_0 = \bigvee_{n=-\infty}^{\infty} V_2^n H = \bigvee_{s=-\infty}^{\infty} V_2(s)H$ , which completes the proof.

#### References

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