

## An inverse problem for ordinary differential equations of a higher order

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**Introduction.** The inverse problem for differential equations, so-called the inverse problem of the Sturm–Liouville type, lies in the settlement of the dependence of the coefficients of the differential equation on the eigenvalues of a suitable problem. This problem has been treated in many papers and has been examined in its various bearings (see [1], [2], [3], [4], [7], [8], [6]). The method of this paper is based on the idea of the paper [1], which was then expanded in the papers [6], [2], [3], [4] and the papers [2], [3], [4], and [6] dealt with partial differential equations, while this paper deals with ordinary differential equations.

### 1. The Green function of an ordinary differential equation

Let us consider the eigenvalues and eigenfunctions for the equation

$$(1) \quad y'' + \lambda y = 0, \quad a < x < b,$$

with boundary conditions

$$(2) \quad \alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are non-negative constants fulfilling the conditions  $\alpha_1^2 + \alpha_2^2 > 0$  and  $\beta_1^2 + \beta_2^2 > 0$ .

Let  $\{\mu_n\}$  and  $\{\varphi_n\}$  be the sequences of eigenvalues and eigenfunctions of problem (1), (2), respectively, and let  $G(x, \xi, \lambda)$  be the Green function of problem (1), (2) with the pole in point  $\xi$ . Let us denote by  $G_0(x, \xi, \lambda)$  the Green function of equation (1) in the interval  $(-\infty, +\infty)$ . Assume that  $\lambda$  is a real negative number and we denote  $\varrho = -\lambda$ . It is known that

$$(3) \quad G_0(x, \xi, -\varrho) = \frac{1}{2\sqrt{\varrho}} e^{-\sqrt{\varrho}|x-\xi|}.$$

Let us put

$$(4) \quad F(x, \xi, \varrho) = G_0(x, \xi, -\varrho) - G(x, \xi, -\varrho).$$

From the definition of the Green function it follows that the function  $F$ , as the function of variable  $x$ , with fixed  $\xi$ , is the function of class  $C^2$  in  $(a, b)$ , and satisfies the equation

$$(5) \quad \frac{\partial^2 F}{\partial x^2} = \varrho F.$$

From the equation (5) it follows that the function  $F$ , as a function of variable  $x$ , attains neither its positive maximum nor its negative minimum in the interval  $(a, b)$ .

Remark 1. In the sequel, the two following cases of boundary conditions (2) will play an important part:

1. a boundary condition of the Dirichlet type, i.e.,  $\alpha_2 = \beta_2 = 0$ ,
2. a boundary condition of the Neumann type, i.e.,  $\alpha_1 = \beta_1 = 0$ .

In these cases the function  $F$ , defined by formula (4), we denote by  $F_1$  and  $F_2$ , respectively.

We shall prove the following lemmas

LEMMA 1. *The function  $F_1$ , defined in the Remark 1, is non-negative in  $[a, b] \times [a, b]$  for every  $\varrho > 0$ .*

Proof. Let  $\xi$  be a fixed point in the interval  $[a, b]$ . Because

$$G(a, \xi, -\varrho) = G(b, \xi, -\varrho) = 0,$$

then  $F_1(a, \xi, \varrho) = G_0(a, \xi, -\varrho)$  and  $F_1(b, \xi, \varrho) = G_0(b, \xi, -\varrho)$ . From the definition of the function  $G_0$  it follows that  $F_1(a, \xi, \varrho) > 0$  and  $F_1(b, \xi, \varrho) > 0$ . Since the function  $F_1$  does not attain its negative minimum in the interval  $(a, b)$ , we have the inequality

$$F_1(x, \xi, \varrho) \geq 0 \quad \text{for each } x \in [a, b].$$

From this and by the symmetry property of the function  $F_1$  with respect to the points  $x, \xi$  follows the inequality

$$(6) \quad \forall x, \xi \in [a, b] \forall \varrho > 0 \quad F_1(x, \xi, \varrho) \geq 0.$$

The inequality (6) is the thesis of Lemma 1.

LEMMA 2. *The function  $F_2$  defined in Remark 1, is non-positive in  $[a, b] \times [a, b]$  for every  $\varrho > 0$ .*

Proof. Let  $\xi$  be a fixed point in the interval  $[a, b]$ . Suppose that the function  $F_2$  has positive values in the interval  $[a, b]$ . From this we get

$$(7) \quad \max_{x \in [a, b]} F_2(x, \xi, \varrho) > 0.$$

Because  $F_2$  is the continuous function in the interval  $[a, b]$ , then exists a point  $x_0 \in [a, b]$ , such that

$$(8) \quad F_2(x_0, \xi, \varrho) = \max_{x \in [a, b]} F_2(x, \xi, \varrho).$$

As we know,  $x_0 \notin (a, b)$  and so  $x_0 = a$  or  $x_0 = b$ . If  $x_0 = a$ , we have

$$(9) \quad \frac{\partial F_2}{\partial x} \Big|_{x=a} = \frac{\partial G_0}{\partial x} \Big|_{x=a} - \frac{\partial G}{\partial x} \Big|_{x=a} = \frac{\partial G_0}{\partial x} \Big|_{x=a} = \frac{1}{2} e^{\sqrt{\varrho}(a-\xi)} > 0.$$

By (8), we get

$$(10) \quad \frac{\partial F_2}{\partial x} \Big|_{x=a} \leq 0,$$

which is a contradiction. Thus  $x_0 \neq a$ . Analogously we prove that  $x_0 \neq b$ . Whence

$$(11) \quad \forall x \in [a, b] F_2(x, \xi, \varrho) \leq 0.$$

By the symmetry property of the function  $F_2$  with respect to the points  $x, \xi$ , and from (11), follows the thesis of Lemma 2.

Let us denote

$$(12) \quad \Phi(x, \xi, \varrho) := \frac{\partial F}{\partial \varrho}(x, \xi, \varrho)$$

and analogously

$$(13) \quad \Phi_i(x, \xi, \varrho) := \frac{\partial F_i}{\partial \varrho}(x, \xi, \varrho), \quad i = 1, 2.$$

for  $x, \xi \in [a, b]$  and  $\varrho > 0$ . Let us observe that for fixed  $\xi \in [a, b]$ , the function  $\Phi$  and  $\Phi_i$ ,  $i = 1, 2$  satisfy the equations

$$(14) \quad \frac{\partial^2 \Phi}{\partial x^2} = \varrho \Phi + F \quad \text{and} \quad \frac{\partial^2 \Phi_i}{\partial x^2} = \varrho \Phi_i + F_i, \quad i = 1, 2.$$

We shall prove the following

**THEOREM 1.** *The function  $\Phi_1$  is non-positive, while the function  $\Phi_2$  is non-negative in  $[a, b] \times [a, b]$  for every  $\varrho > 0$ .*

*Proof.* Let  $\xi$  be a fixed point in the interval  $[a, b]$ . By Lemma 1 the function  $F_1$  is non-negative in the interval  $[a, b]$ . Thus from (14) it follows that the function  $\Phi_1$  does not attain its positive maximum in the interval  $(a, b)$ . Because

$$\frac{\partial G}{\partial \varrho}(x, \xi, -\varrho) = 0$$

for  $x = a$  and  $x = b$ , when

$$(15) \quad \Phi_1(x, \xi, \varrho) = \frac{\partial G_0}{\partial \varrho}(x, \xi, -\varrho) = -\frac{1}{4} e^{-\sqrt{\varrho}|x-\xi|} (\varrho^{-\frac{1}{2}} + |x-\xi| \varrho^{-1}),$$

for  $x = a$  and  $x = b$ . From the equality (15) it follows that

$$(16) \quad \Phi_1(a, \xi, \varrho) < 0 \quad \text{and} \quad \Phi_1(b, \xi, \varrho) < 0.$$

The first part of the thesis of Theorem 1 follows from the inequalities (16), and the previous remarks.

The proof of the second part of Theorem 1 is quite similar (see also the proof of Lemma 2), and is omitted.

Remark 2. The functions  $\Phi$ ,  $\Phi_1$  and  $\Phi_2$  defined above satisfy the following inequality

$$(17) \quad \forall x, \xi \in [a, b] \forall \varrho > 0 \quad \Phi_1(x, \xi, \varrho) \leq \Phi(x, \xi, \varrho) \leq \Phi_2(x, \xi, \varrho)$$

(cf. Theorem 1 of paper [2]).

## • 2. Further properties of the Green function of problems (1), (2)

As we know, the Green function of problems (1), (2) with the pole at the point  $\xi$ , is defined as the solution of the equation

$$(18) \quad y'' + \lambda y = \delta(\xi - x),$$

satisfying the boundary conditions (2). The Green function of problem (1), (2) with the pole at the point  $\xi$ , will be denoted by  $G(x, \xi, \lambda)$ , analogously to 1.

We shall prove the following

LEMMA 3. If  $\lambda$  and  $\lambda_0$  are any fixed negative numbers, then for all points  $x, y \in (a, b)$ , we have the equality

$$(19) \quad (\lambda - \lambda_0) \int_a^b G(t, x, \lambda) G(t, y, \lambda_0) dt = G(x, y, \lambda) - G(x, y, \lambda_0).$$

Proof. Using the integration-by-part formula and using the boundary conditions (2) for function  $G$ , we get

$$(20) \quad \int_a^b \left[ \frac{\partial^2 G}{\partial t^2}(t, x, \lambda) G(t, y, \lambda_0) - \frac{\partial^2 G}{\partial t^2}(t, y, \lambda_0) G(t, x, \lambda) \right] dt = 0.$$

From this we have

$$\begin{aligned} & (\lambda - \lambda_0) \int_a^b G(t, x, \lambda) G(t, y, \lambda_0) dt \\ &= \int_a^b \left\{ \left[ \frac{\partial^2 G}{\partial t^2}(t, x, \lambda) + \lambda G(t, x, \lambda) \right] G(t, y, \lambda_0) - \right. \\ & \quad \left. - \left[ \frac{\partial^2 G}{\partial t^2}(t, y, \lambda_0) + \lambda_0 G(t, y, \lambda_0) \right] G(t, x, \lambda) \right\} dt \\ &= \int_a^b [\delta(x-t)G(t, y, \lambda_0) - \delta(y-t)G(t, x, \lambda)] dt = -G(x, y, \lambda_0) + G(y, x, \lambda) \\ &= G(x, y, \lambda) - G(x, y, \lambda_0). \end{aligned}$$

The proof of Lemma 3 is complete.

THEOREM 2. For each  $x \in (a, b)$  and for every  $\varrho > 0$ , we have the following equality

$$(21) \quad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + \varrho)^2} = \frac{1}{4} \varrho^{-\frac{3}{2}} + \Phi(x, x, \varrho),$$

where  $\Phi$  is the function defined by formula (12).

Proof. Let  $\lambda$  be a fixed real number, such that  $\lambda \neq \mu_n, n = 1, 2, \dots$  and let  $x \in (a, b)$ . We have

$$\begin{aligned} & (\mu_n - \lambda) \int_a^b G(x, y, \lambda) \varphi_n(y) dy \\ &= \int_a^b \left\{ [\varphi_n''(y) + \mu_n \varphi_n(y)] G(x, y, \lambda) - \left[ \frac{\partial^2 G}{\partial y^2}(x, y, \lambda) + \lambda G(x, y, \lambda) \right] \varphi_n(y) \right\} dy - \\ & - \int_a^b \left[ \varphi_n''(y) G(x, y, \lambda) - \varphi_n(y) \frac{\partial^2 G}{\partial y^2}(x, y, \lambda) \right] dy = - \int_a^b \delta(x-y) \varphi_n(y) dy = \varphi_n(x), \end{aligned}$$

because

$$\varphi_n''(y) + \mu_n \varphi_n(y) = 0 \quad \text{and} \quad \int_a^b \left[ \varphi_n''(y) G(x, y, \lambda) - \varphi_n(y) \frac{\partial^2 G}{\partial y^2}(x, y, \lambda) \right] dy = 0.$$

Whence

$$(22) \quad \int_a^b G(x, y, \lambda) \varphi_n(y) dy = \frac{\varphi_n(x)}{\mu_n - \lambda}, \quad n = 1, 2, \dots$$

The equalities (22) denote that for fixed  $x \in (a, b)$ ,  $\left\{ \frac{\varphi_n(x)}{\mu_n - \lambda} \right\}$  is the sequence of the Fourier coefficients of the Green function  $G(x, y, \lambda)$  with respect to the orthonormal system  $\{\varphi_n\}$ . By the completeness of the sequence  $\{\varphi_n\}$ , and by the Parseval identity from (19), we get

$$(23) \quad (\lambda - \lambda_0) \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(y)}{(\mu_n - \lambda_0)(\mu_n - \lambda)} = G(x, y, \lambda) - G(x, y, \lambda_0).$$

Putting in equality (23)  $\varrho = -\lambda$  and  $\varrho_0 = -\lambda_0$  and using formula (4), we have

$$(24) \quad \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(y)}{(\mu_n + \varrho_0)(\mu_n + \varrho)} = \frac{G_0(x, y, -\varrho) - G_0(x, y, -\varrho_0)}{\varrho_0 - \varrho} + \frac{F(x, y, \varrho_0) - F(x, y, \varrho)}{\varrho_0 - \varrho}.$$

Let  $\varrho_0 \rightarrow \varrho$  in (24). Then

$$(25) \quad \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{(\mu_n + \varrho)^2} = -\frac{\partial G_0}{\partial \varrho}(x, y, -\varrho) + \Phi(x, y, \varrho).$$

Using the equality (3), and the continuity of functions  $\partial G_0/\partial \varrho$ ,  $\Phi$  and  $\varphi_n$ ,  $n = 1, 2, \dots$ , from (25) while  $y \rightarrow x$ , we get (21).

**THEOREM 3.** *The function  $\Phi$  in the formula (21), satisfies the following condition*

$$(26) \quad \int_a^b |\Phi(x, x, \varrho)| dx = o(\varrho^{-\frac{3}{2}}) \quad \text{for } \varrho \rightarrow +\infty.$$

*Proof.* Since  $\Phi$  is a continuous function in the interval  $[a, b]$ , whence by Dini's theorem, the series on the left hand of the formula (21) is uniformly convergent in the interval  $[a, b]$ . Integrating the equality (21) into the interval  $[a, b]$ , we get

$$(27) \quad \sum_{n=1}^{\infty} \frac{1}{(\mu_n + \varrho)^2} = \frac{1}{4}(b-a)\varrho^{-\frac{3}{2}} + \int_a^b \Phi(x, x, \varrho) dx.$$

On the other hand (cf. [2])

$$(28) \quad \sum_{n=1}^{\infty} \frac{1}{(\mu_n + \varrho)^2} = \frac{1}{4}(b-a)\varrho^{-\frac{3}{2}} + o(\varrho^{-\frac{3}{2}}) \quad \text{for } \varrho \rightarrow +\infty.$$

From the equalities (27) and (28) follows

$$(29) \quad \int_a^b \Phi(x, x, \varrho) dx = o(\varrho^{-\frac{3}{2}}) \quad \text{for } \varrho \rightarrow +\infty.$$

The equality (29) is true in particular for the function  $\Phi_1$  and  $\Phi_2$ . From the inequality (17), we get

$$(30) \quad \forall x \in [a, b] \forall \varrho > 0 \quad \Phi_1(x, x, \varrho) \leq \Phi(x, x, \varrho) \leq \Phi_2(x, x, \varrho).$$

Because  $\Phi_1(x, x, \varrho) \leq 0$  and  $\Phi_2(x, x, \varrho) \geq 0$  for every  $x \in [a, b]$  and  $\varrho > 0$ , therefore from (30) it follows that

$$(31) \quad \Phi_1(x, x, \varrho) - \Phi_2(x, x, \varrho) \leq \Phi(x, x, \varrho) \leq \Phi_2(x, x, \varrho) - \Phi_1(x, x, \varrho)$$

for every  $x \in [a, b]$  and  $\varrho > 0$ . The inequality (31) may be written in the form

$$(32) \quad \forall x \in [a, b] \forall \varrho > 0 \quad |\Phi(x, x, \varrho)| \leq \Phi_2(x, x, \varrho) - \Phi_1(x, x, \varrho).$$

From (32) and (29) applying to the functions  $\Phi_1$  and  $\Phi_2$ , follows the equality (26). The proof of Theorem 3 is complete.

The results of Theorems 2 and 3 may be expressed in the following

COROLLARY 1. The eigenvalues and eigenfunctions of problem (1), (2) are related to the equality

$$(33) \quad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + \varrho)^2} = \frac{1}{4} \varrho^{-\frac{3}{2}} + \Phi(x, x, \varrho),$$

where

$$(34) \quad \int_a^b |\Phi(x, x, \varrho)| dx = o(\varrho^{-\frac{3}{2}}) \quad \text{for } \varrho \rightarrow +\infty.$$

### 3. The eigenvalues and eigenfunctions of a differential equation of a higher order

In 2, using the Green function of the problem (1), (2), we proved formula (33). In this section we shall generalize formula (33) in the case of an ordinary equation of a higher order.

$$(35) \quad (-1)^m y^{(2m)} - \lambda y = 0$$

with boundary conditions

$$(36) \quad \alpha_1 y^{(2\nu)}(a) - \alpha_2 y^{(2\nu+1)}(a) = 0, \quad \beta_1 y^{(2\nu)}(b) + \beta_2 y^{(2\nu+1)}(b) = 0,$$

where  $\nu = 0, 1, \dots, m-1$ ,  $m \geq 1$ ,  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are constants occurring in boundary conditions (2).

Remark 3. The method used in 2 to prove the formulas (33) and (34) cannot be used in the case of problem (35), (36) if  $m > 1$  because in this case for equation (35) the maximum principle is not true.

Therefore in this section we must use another method, used in the author's previous papers, in particular in paper [2].

As we know, the eigenvalues of problem (35), (36) are the  $m$ -th power of the eigenvalues of problem (1), (2), while the eigenfunctions of these problems are identical.

Let us denote by  $\{\mu_n\}$  and  $\{\varphi_n\}$  the sequences of the eigenvalues and eigenfunctions of problem (35), (36), respectively. We obtain

$$(37) \quad \mu_n = \left( \frac{\pi n}{b-a} \right)^{2m} + o(n^{2m}).$$

The eigenfunctions of problem (35), (36) belong to the class  $C^{2m}([a, b])$  and together are bounded, i.e.,

$$(38) \quad \exists M > 0 \quad \forall x \in [a, b] \quad \forall n \in N \quad |\varphi_n(x)| \leq M.$$

By formula (7) from paper [4] and by (37), we get

$$(39) \quad \sum_{n=1}^{\infty} \frac{1}{(\mu_n + \varrho)^2} = A \varrho^{\frac{1}{2m}-2} + o(\varrho^{\frac{1}{2m}-2}) \quad \text{for } \varrho \rightarrow +\infty,$$

where

$$A = \frac{b-a}{\pi} \Gamma\left(1 + \frac{1}{2m}\right) \Gamma\left(2 - \frac{1}{2m}\right).$$

From the equality (39), in virtue of (38), it follows that the series

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + \varrho)^2}$$

is uniformly convergent in  $[a, b]$ , for every  $\varrho > 0$ . Therefore denoting by

$$(40) \quad F(x, \varrho) := \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + \varrho)^2},$$

we see that the function  $F$  is continuous for  $x \in [a, b]$  and  $\varrho > 0$ .

Integrating the identity (40) over the interval  $[a, b]$ , we get

$$(41) \quad \sum_{n=1}^{\infty} \frac{1}{(\mu_n + \varrho)^2} = \int_a^b F(x, \varrho) dx \quad \text{for } \varrho > 0.$$

From (39) and (41) it follows that

$$(42) \quad \int_a^b F(x, \varrho) dx = A \varrho^{\frac{1}{2m}-2} + o(\varrho^{\frac{1}{2m}-2}), \quad \text{for } \varrho \rightarrow +\infty.$$

Presenting the function  $F$  in the form

$$(43) \quad F(x, \varrho) = \frac{A}{b-a} \varrho^{\frac{1}{2m}-2} + \Phi(x, \varrho),$$

we have the equality

$$(44) \quad \int_a^b \Phi(x, \varrho) dx = o(\varrho^{\frac{1}{2m}-2}), \quad \text{for } \varrho \rightarrow +\infty,$$

where  $\Phi$  is a continuous function for  $x \in [a, b]$  and  $\varrho > 0$ . From the equalities (40) and (43) we obtain

$$(45) \quad \sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + \varrho)^2} = \frac{A}{b-a} \varrho^{\frac{1}{2m}-2} + \Phi(x, \varrho),$$

where the function  $\Phi$  satisfies the condition (44).

Now we shall prove that the function  $\Phi$  satisfies the condition

$$(46) \quad \int_a^b |\Phi(x, \varrho)| dx = o(\varrho^{\frac{1}{2m}-2}), \quad \text{for } \varrho \rightarrow +\infty.$$

For this purpose we use the following asymptotic formula

$$(47) \quad \sum_{v=1}^n \varphi_v^2(x) = \frac{1}{\pi} \sqrt{\lambda_n} + o(1).$$

This formula (47) is a particular case of formula (18.7.4) from monograph [10]. From formula (47) it follows that for the arbitrary fixed points  $x_0, y_0 \in (a, b)$ , the sequence of the partial sum of the series

$$\sum_{n=1}^{\infty} [\varphi_n^2(x_0) - \varphi_n^2(y_0)]$$

is a bounded sequence. Let us denote

$$(48) \quad a_n = \varphi_n^2(x_0) - \varphi_n^2(y_0), \quad n \in N.$$

From this and from Abel's theorem it follows that the series

$$(49) \quad \sum_{n=1}^{\infty} \frac{a_n}{\mu_n^\alpha}$$

is convergent for arbitrary  $\alpha > 0$ .

We shall prove the following

**THEOREM 4.** *If  $\{\mu_n\}$  and  $\{\varphi_n\}$  are sequences of the eigenvalues and eigenfunctions of problem (35), (36), respectively, then the series*

$$(50) \quad \sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^2} \varrho^{2 - \frac{1}{2m}}$$

is uniformly convergent in  $\varrho$ , for  $\varrho > 0$ , where  $a_n$  is defined by (48).

**Proof.** First, we observe that the series

$$(51) \quad \sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^{\frac{1}{2m}}}$$

is uniformly convergent in  $\varrho$ , for  $\varrho > 0$ . Indeed, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^{\frac{1}{2m}}} = \sum_{n=1}^{\infty} \frac{a_n}{\mu_n^{\frac{1}{2m}}} \left( \frac{\mu_n}{\mu_n + \varrho} \right)^{\frac{1}{2m}}$$

Since  $\left\{ \left( \frac{\mu_n}{\mu_n + \varrho} \right)^{\frac{1}{2m}} \right\}$  is the increasing sequence for each fixed  $\varrho > 0$  and the uniformly bounded sequence, while the series (49) is convergent, then by Abel's theorem the series (51) is uniformly convergent in  $\varrho$ , for  $\varrho > 0$ .

Converting the series (50) as follows

$$\sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^2} \varrho^{2 - \frac{1}{2m}} = \sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^{\frac{1}{2m}}} \left( \frac{\varrho}{\mu_n + \varrho} \right)^{2 - \frac{1}{2m}},$$

and again using Abel's theorem, we get the thesis of Theorem 4.

**THEOREM 5.** *The function  $\Phi$  occurring in the formula (45) satisfies the condition (46).*

**Proof.** First, let us observe that from condition (38), there follows the inequality

$$\forall x \in [a, b] \forall \varrho > 0 \quad |\Phi(x, \varrho)| \leq M^2 \sum_{n=1}^{\infty} \frac{1}{(\mu_n + \varrho)^2} + \frac{A}{b-a} \varrho^{\frac{1}{2m} - 2},$$

and from this, by (39), we have

$$(52) \quad \forall x \in [a, b] \forall \varrho > 0 \quad |\Phi(x, \varrho)| \varrho^{2 - \frac{1}{2m}} \leq C,$$

where  $C > 0$  is a constant independent of  $x$  and of  $\varrho$ . Hence

$$(53) \quad \forall x \in [a, b] \limsup_{\varrho \rightarrow \infty} |\Phi(x, \varrho)| \varrho^{2 - \frac{1}{2m}} \leq C.$$

Let  $x_0 \in (a, b)$  be a fixed point. We denote by

$$g_1 = \limsup_{\varrho \rightarrow \infty} \Phi(x_0, \varrho) \varrho^{2 - \frac{1}{2m}}$$

and

$$g_2 = \liminf_{\varrho \rightarrow \infty} \Phi(x_0, \varrho) \varrho^{2 - \frac{1}{2m}}.$$

On the other hand, by equality (45), we obtain

$$\Phi(x_0, \varrho) - \Phi(y_0, \varrho) = \sum_{n=1}^{\infty} \frac{a_n}{(\mu_n + \varrho)^2},$$

where  $a_n$ ,  $n \in N$ , are defined by formula (48).

From this, in virtue of Theorem 4, it follows that

$$\limsup_{\varrho \rightarrow \infty} \Phi(x, \varrho) \varrho^{2 - \frac{1}{2m}} = g_1 \quad \text{and} \quad \liminf_{\varrho \rightarrow \infty} \Phi(x, \varrho) \varrho^{2 - \frac{1}{2m}} = g_2,$$

for every  $x \in (a, b)$ .

Using Lebesgue's theorem, we have

$$0 = \limsup_{\varrho \rightarrow \infty} \varrho^{2 - \frac{1}{2m}} \int_a^b \Phi(x, \varrho) dx = \int_a^b [\limsup_{\varrho \rightarrow \infty} \Phi(x, \varrho) \varrho^{2 - \frac{1}{2m}}] dx = g_1(b-a)$$

and

$$0 = \liminf_{\varrho \rightarrow \infty} \varrho^{2-\frac{1}{2m}} \int_a^b \Phi(x, \varrho) dx = \int_a^b [\liminf_{\varrho \rightarrow \infty} \Phi(x, \varrho) \varrho^{2-\frac{1}{2m}}] dx = g_2(b-a).$$

From these equalities it follows that

$$(54) \quad \forall x \in (a, b) \lim_{\varrho \rightarrow \infty} \Phi(x, \varrho) \varrho^{2-\frac{1}{2m}} = 0.$$

From the equality (54) there directly follows (46), i.e., the thesis of Theorem 5.

#### 4. Generalization an Ambarzumian's theorem

First of all in this section we shall state one theorem on the trace of a linear operator in a separable Hilbert space.

Let  $H$  be a real separable Hilbert space and let  $T, V$  be the self-adjoint linear operators defined in  $H$ , and with their values in  $H$ . We assume that  $V$  is a bounded operator. We denote by  $T_\varrho = T + \varrho I$ , where  $\varrho$  is a real positive number and  $I$  is the identity operator in  $H$ . We also assume that  $T_\varrho^{-1}$  and  $(T_\varrho + V)^{-1}$  are completely continuous operators, for a sufficiently large number  $\varrho$ . Let us denote by  $\{\mu_n\}$  and  $\{\lambda_n\}$  the increasing sequences of all eigenvalues of  $T$  and  $T + V$ , respectively.

On these assumptions we have the following

LEMMA 4. *If*

1° *the sequence  $\{\mu_n\}$  satisfies the condition*

$$(55) \quad \mu_n = Cn^\alpha + o(n^\alpha),$$

where  $C$  and  $\alpha$  are positive constants independent of  $n$ ,

2°  *$k$  is a natural number such that  $k \geq \frac{1}{\alpha}$ ,*

3° *the series*

$$\sum_{n=1}^{\infty} \frac{|\lambda_n - \mu_n|}{n}$$

is convergent, then

$$(56) \quad \lim_{\varrho \rightarrow +\infty} \varrho^{k+1-\frac{1}{\alpha}} S(VT_\varrho^{-k-1}) = 0,$$

where  $S(A)$  denotes the trace of the operator  $A$ .

The proof of Lemma 4 is omitted, as it is quite similar to the proof of Theorem 2 in paper [4].

Besides the problem (35), (36) defined in 3, we now consider the following equation

$$(57) \quad (-1)^m y^{(2m)} - [\lambda - q(x)]y = 0, \quad x \in (a, b)$$

with the boundary conditions (36). Here  $q$  is a continuous functions defined in the interval  $[a, b]$ .

Let  $\{\mu_n\}$  and  $\{\varphi_n\}$  denote the sequences of the eigenvalues and eigenfunctions of problem (35), (36), and  $\{\lambda_n\}$  and  $\{\psi_n\}$  these sequences for the problem (57), (36).

We shall prove the following

**THEOREM 6.** *If the series  $\sum_{n=1}^{\infty} \frac{|\lambda_n - \mu_n|}{n}$  is convergent, then*

$$(58) \quad \int_a^b q(x) dx = 0.$$

**Proof.** Let  $H = L_2([a, b])$  and let  $T$  be a self-adjoint operator on  $H$  which is a Friedrichs expansion of the differential operator  $(-1)^m y^{(2m)}$ ,  $m \geq 1$ , defined in the subspace of  $H$ , which is composed of the functions of class  $C^{2m}([a, b])$  satisfying the conditions (36), while  $V$  is the multiplier operator  $q$ . From the equality (37) and from the assumptions of Theorem 6, it follows that the operators  $T$  and  $V$  satisfy the assumptions of Lemma 4, with constant  $\alpha = 2m$ . Since  $m \geq 1$ , then we may take  $k = 1$  in the condition (56). Then we have

$$(59) \quad \lim_{q \rightarrow \infty} q^{2 - \frac{1}{2m}} S(VT_q^{-2}) = 0.$$

We express the trace of the operator  $VT_q^{-2}$  by the sequence  $\{\varphi_n\}$ . We get

$$(60) \quad S(VT_q^{-2}) = \sum_{n=1}^{\infty} \frac{(V\varphi_n, \varphi_n)}{(\mu_n + q)^2}.$$

By the definition of the operator  $V$  and by the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{\varphi_n^2(x)}{(\mu_n + q)^2}$$

with respect to  $x \in [a, b]$ , the equality (60) may be written in the form

$$(61) \quad S(VT_q^{-2}) = \left( q, \sum_{n=1}^{\infty} \frac{\varphi_n^2}{(\mu_n + q)^2} \right).$$

Using the formula (45), we get

$$(62) \quad S(VT_q^{-2}) = \frac{A}{b-a}(q, 1)q^{\frac{1}{2m}-2} + (q, \Phi).$$

Since

$$|(q, \Phi)| = \left| \int_a^b q(x) \Phi(x, \varrho) dx \right| \leq \max_{a \leq x \leq b} |q(x)| \int_a^b |\Phi(x, \varrho)| dx,$$

we have by (46)

$$(63) \quad (q, \Phi) = o(\varrho^{\frac{1}{2m}-2}).$$

In virtue of (59), (62) and (63) we have

$$(64) \quad \lim_{\varrho \rightarrow \infty} \left\{ \frac{A}{b-a} (q, 1) + o(1) \right\} = 0.$$

The equality (64) is possible only when

$$(65) \quad (q, 1) = 0.$$

It is obvious that (65) is equivalent to (58), and this concludes the proof of Theorem 6.

**THEOREM 7.** *Under the assumptions of Theorem 6 and if the function  $q$  also satisfies the condition*

$$(66) \quad \forall \varphi \in C_0^1([a, b]) \quad \int_a^b [\varphi'^2(x) + q(x)\varphi^2(x)] dx \geq 0,$$

then  $q(x) = 0$  for every  $x \in [a, b]$ , i.e., the problems (35), (36) and (57), (36) are identical.

**Proof.** From Theorem 6 it follows that the function  $q$  satisfies the condition (58). On the other hand, from (66), by virtue of the variational definition of the eigenvalues (cf. [5]), it follows that the first eigenvalue of the equation

$$(67) \quad y'' + [\lambda - q(x)]y = 0$$

with boundary condition  $y'(a) = y'(b) = 0$ , is a non-negative number, because

$$(68) \quad \lambda_1 = \min_{\varphi \in C_0^1([a, b])} \int_a^b [\varphi'^2(x) + q(x)\varphi^2(x)] dx.$$

On the other hand, by (58), the function  $\varphi_1(x) = 1/\sqrt{b-a}$  realizes the minimum (68), which is equal to zero. Therefore, the first eigenvalue of the equation (67), with the boundary condition  $y'(a) = y'(b) = 0$  is  $\lambda_1 = 0$ , which corresponds to the first eigenfunction  $\varphi_1(x) = \frac{1}{\sqrt{b-a}}$ . This means that this function  $\varphi_1$  satisfies the equation (67) for  $\lambda = 0$ .

Hence

$$\forall x \in (a, b) q(x) \frac{1}{\sqrt{b-a}} = 0,$$

i.e.,  $q(x) = 0$  for every  $x \in (a, b)$ . The proof of Theorem 7 is complete.

**Remark 4.** The condition (66) is fulfilled if  $q$  is a non-negative function in the interval  $[a, b]$ , but not only in this case.

