

An inverse problem for ordinary differential equations of a higher order

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Introduction. The inverse problem for differential equations, so-called the inverse problem of the Sturm–Liouville type, lies in the settlement of the dependence of the coefficients of the differential equation on the eigenvalues of a suitable problem. This problem has been treated in many papers and has been examined in its various bearings (see [1], [2], [3], [4], [7], [8], [6]). The method of this paper is based on the idea of the paper [1], which was then expanded in the papers [6], [2], [3], [4] and the papers [2], [3], [4], and [6] dealt with partial differential equations, while this paper deals with ordinary differential equations.

1. The Green function of an ordinary differential equation

Let us consider the eigenvalues and eigenfunctions for the equation

$$(1) \quad y'' + \lambda y = 0, \quad a < x < b,$$

with boundary conditions

$$(2) \quad \alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are non-negative constants fulfilling the conditions $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$.

Let $\{\mu_n\}$ and $\{\varphi_n\}$ be the sequences of eigenvalues and eigenfunctions of problem (1), (2), respectively, and let $G(x, \xi, \lambda)$ be the Green function of problem (1), (2) with the pole in point ξ . Let us denote by $G_0(x, \xi, \lambda)$ the Green function of equation (1) in the interval $(-\infty, +\infty)$. Assume that λ is a real negative number and we denote $\varrho = -\lambda$. It is known that

$$(3) \quad G_0(x, \xi, -\varrho) = \frac{1}{2\sqrt{\varrho}} e^{-\sqrt{\varrho}|x-\xi|}.$$

Let us put

$$(4) \quad F(x, \xi, \varrho) = G_0(x, \xi, -\varrho) - G(x, \xi, -\varrho).$$

From the definition of the Green function it follows that the function F , as the function of variable x , with fixed ξ , is the function of class C^2 in (a, b) , and satisfies the equation

$$(5) \quad \frac{\partial^2 F}{\partial x^2} = \varrho F.$$

From the equation (5) it follows that the function F , as a function of variable x , attains neither its positive maximum nor its negative minimum in the interval (a, b) .

Remark 1. In the sequel, the two following cases of boundary conditions (2) will play an important part:

1. a boundary condition of the Dirichlet type, i.e., $\alpha_2 = \beta_2 = 0$,
2. a boundary condition of the Neumann type, i.e., $\alpha_1 = \beta_1 = 0$.

In these cases the function F , defined by formula (4), we denote by F_1 and F_2 , respectively.

We shall prove the following lemmas

LEMMA 1. *The function F_1 , defined in the Remark 1, is non-negative in $[a, b] \times [a, b]$ for every $\varrho > 0$.*

Proof. Let ξ be a fixed point in the interval $[a, b]$. Because

$$G(a, \xi, -\varrho) = G(b, \xi, -\varrho) = 0,$$

then $F_1(a, \xi, \varrho) = G_0(a, \xi, -\varrho)$ and $F_1(b, \xi, \varrho) = G_0(b, \xi, -\varrho)$. From the definition of the function G_0 it follows that $F_1(a, \xi, \varrho) > 0$ and $F_1(b, \xi, \varrho) > 0$. Since the function F_1 does not attain its negative minimum in the interval (a, b) , we have the inequality

$$F_1(x, \xi, \varrho) \geq 0 \quad \text{for each } x \in [a, b].$$

From this and by the symmetry property of the function F_1 with respect to the points x, ξ follows the inequality

$$(6) \quad \forall x, \xi \in [a, b] \forall \varrho > 0 \quad F_1(x, \xi, \varrho) \geq 0.$$

The inequality (6) is the thesis of Lemma 1.

LEMMA 2. *The function F_2 defined in Remark 1, is non-positive in $[a, b] \times [a, b]$ for every $\varrho > 0$.*

Proof. Let ξ be a fixed point in the interval $[a, b]$. Suppose that the function F_2 has positive values in the interval $[a, b]$. From this we get

$$(7) \quad \max_{x \in [a, b]} F_2(x, \xi, \varrho) > 0.$$

Because F_2 is the continuous function in the interval $[a, b]$, then exists a point $x_0 \in [a, b]$, such that

$$(8) \quad F_2(x_0, \xi, \varrho) = \max_{x \in [a, b]} F_2(x, \xi, \varrho).$$

