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On some characterization of non-fixed points of mappings

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Given a mapping f of which the domain is an abstract set without any algebraic or topological structure, we shall characterize below a set of its non-fixed points. The possibility of such a characterization was suggested by Professor Z. Opial.

Let S be any set and f any mapping from S. A. Abian in [1] characterized a set of non-fixed points of f in the following manner.

Theorem 1. (A. Abian [1]) *Let f be any mapping from S, $\overline{\overline{S}} = \aleph_0$ and for every $E \subset S$ such that $\overline{\overline{E}} = \aleph_0$ there is $E \cap f(E) \neq \emptyset$. Then*

$$\overline{\overline{N}}_f < \aleph_0 .$$

Where $N_f = \{x \in S: f(x) \neq x\}$ and for any set A by $\overline{\overline{A}}$ is denoted its cardinality.

Taking into consideration Abian's proof it is easy to generalize this theorem in the following way.

Theorem 2. *Let f be any mapping from S, $\overline{\overline{S}} = \mathfrak{t} \geqslant \aleph_0$ and for every $E \subset S$ such that $\overline{\overline{E}} = \mathfrak{t}$ there is $E \cap f(E) \neq \emptyset$. Then*

$$\overline{\overline{N}}_f < \mathfrak{t} .$$

Now we shall prove some generalization of this statement.

Theorem 3. *Let f be any mapping from S, $\overline{\overline{S}} = \mathfrak{t}$ and $\mathfrak{t} \geqslant \mathfrak{n} \geqslant \aleph_0$. Then the two following conditions are equivalent:*

1. *for every $E \subset S$ such that $\overline{\overline{E}} = \mathfrak{n}$ there is $E \cap f(E) \neq \emptyset$,*
2. $\overline{\overline{N}}_f < \mathfrak{n}$.

Proof. For the proof of sufficiency (1 implies 2) let us suppose on the contrary that $\overline{\overline{N}}_f \nless \mathfrak{n}$. Now we shall consider a mapping $f|N_Y$ i.e. the restriction of f to the set N_f and two possible cases.

1^0 $\overline{\overline{N}}_f = \mathfrak{n}$.

By the assumptions we have for every $E \subset N_f$ such that $\overline{\overline{E}} = \mathfrak{n}$ (of course such a set exists) $E \cap f(E) \neq \emptyset$. Let N'_f denote the set of all non-fixed points of $f|N_f$. Following

Theorem 2 (an extended version of Abian's Theorem), we have

$$\overline{N}'_f < \overline{N}_f$$

which contradicts the fact that $N'_f = N_f$.

2^0 $\overline{N}_f > \mathfrak{n}$.

For every subset $E \subset N_f$ such that $\overline{E} = \overline{N}_f$ there exists $E' \subset E$ for which $\overline{E}' = \mathfrak{n}$. By the assumptions we have $E' \cap f(E) \neq \emptyset$. Clearly this implies that $E \cap f(E) \neq \emptyset$. By Theorem 2 it follows that

$$\overline{N}'_f < \overline{N}_f$$

contradicting the equality $N'_f = N_f$.

The necessity (2 implies 1) of the condition $\overline{N}_f < n$ is obvious. Let us assume on the contrary that there exists a set $E \subset S$ such that $\overline{E} = \mathfrak{n}$ and $E \cap f(E) = \emptyset$. Of course $E \subset N_f$. This inclusion contradicts the fact that $\overline{N}_f < \mathfrak{n}$. Hence the proof is complete.

Remarks:

1^0 under the assumptions of Theorem 3 the extended version of Abian's Theorem implies only $\overline{N}_f < \mathfrak{k}$,

2^0 if $\mathfrak{k} = \mathfrak{n} = \aleph_0$ then Theorem 3 is reduced to the Abian Theorem and if $\mathfrak{k} = \mathfrak{n} \geqslant \aleph_0$ we have Theorem 2.

References

[1] A. Abian, *A Fixed-Point Theorem for Mappings*, J. Math. Anal. Appl. 24, (1968).