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DCXXIII PRACE MATEMATYCZNE ZESZYT 23 1982

The Whitney condition for sub-analytic sets

by Jacek Stasica

The aim of this paper is to prove that every sub-analytic compact set fulfils Whitney condition. It generalizes the known result for semi-analytic sets [5].

In order to obtain the main theorem we construct a partition of a sub-analytic bounded set in a finite dimensional vector space on a finite number of graphs of analytic functions with bounded differentials.

1⁰. At this point we give definitions and properties of sub-analytic sets useful in the proof of the main theorem.

Let X be a real n-dimensional Euclidean vector space.

Definition 1. A closed subset $E \subset X$ *fulfils the Whitney condition* if for every $a \in E$ there is a neighbourhood U of a and there are constants $M>0$, $\sigma>0$ such that every two points $x, y \in U \cap E$ may be joined by an arc contained in E of length $\leqslant M|x-y|^{\sigma}$.

Definition 2. Subsets A and B are *regularly separated* at $c \in X$ if there is a neighbourhood U of c and there are constants $C>0$, $\alpha>0$ such that $\varrho(x, A)+\varrho(x, B) \geqslant C\varrho(x, A \cap B)^{\alpha}$ for $x \in U$ ($\varrho(x, A)$ denotes the distance between x and A).

A, B are *regularly separated* if they are regularly separated at each point of X.

Notice that when A and B are compact then they are regularly separated if they are regularly separated at each point of $A \cap B$.

Definition 3. A subset $A \subset X$ is *semi-analytic* (*semi-algebraic*) if for every $a \in X$ there is a neighbourhood U of a and there are analytic functions (polynomials) g_{ij}, f_i defined in U such that

$$A \cap U = \bigcup_{i=1}^{r} \Big(\bigcap_{j=1}^{s} \{g_{ij}>0\} \cap \{f_i = 0\}\Big).$$

Definition 4. A bounded subset $E \subset X$ is called *sub-analytic* if there exists a semi-analytic bounded set A contained in some finite dimensional vector space Y such that $E = \Pi(A)$, where $\Pi\colon X \times Y \to X$ is the natural projection.

Definition 5. A germ is called *sub-analytic* if it is the germ of a bounded sub-analytic subset.

Notice that a bounded subset is sub-analytic if and only if its germ at each point of X is sub-analytic. So we may extend definition 4:

Definition 4′. A subset $E \subset X$ is *sub-analytic* if for each $x \in X$ the germ of E at x is sub-analytic.

It follows immediately from this definition that the union of a locally finite family and the intersection of a finite family of sub-analytic sets is a sub-analytic set.

The complement of a sub-analytic set is a sub-analytic set. The proof of this non-trivial result may be found in [2], [3].

The closure of a sub-analytic set is a sub-analytic set $\big($in fact if $V \cap E = \Pi(A)$ with V open and A bounded then $V \cap \bar{E} = \Pi(\bar{A} \cap (V \times Y))\big)$.

The interior of a sub-analytic set is a sub-analytic set $\big(\operatorname{int} E = X \setminus (\overline{X \setminus E})\big)$.

The family of connected components of a sub-analytic set is locally finite (if S is a connected component of E and $V \cap E = \Pi(A)$, so $V \cap S = \bigcup \Pi(S_i)$, where S_i are connected components of A). In particular a bounded sub-analytic set consists of a finite number of connected components.

The Cartesian product of sub-analytic sets is a sub-analytic set.

Every bounded (compact) sub-analytic set is a projection of a bounded (compact) semi-analytic set.

Let $\Pi\colon X \times Y \to X$ be the natural projection. Then if $E \subset X \times Y$ is a sub-analytic set and for each bounded $A \subset X$, $\Pi^{-1}(A)$ is bounded then $\Pi(E)$ is a sub-analytic set.

Definition 6. Let X, Y be finite dimensional vector spaces. A mapping $f\colon A \to Y$, $A \subset X$, is called *sub-analytic* if

$$\hat{f} = \{(x, f(x))\colon x \in A\} \text{ is a sub-analytic subset of } X \times Y.$$

If K, L, P are sub-analytic bounded subsets of finite dimensional vector space X,, Y, Z respectively and $f\colon K \to L$, $g\colon L \to P$ are sub-analytic mappings, then $g \circ f$ is also sub-analytic mapping $\big($for the proof let $\Pi\colon X \times Y \times Z \to X \times Z$ be the natural projection, then we have $\widehat{g \circ f} = \Pi((\hat{f} \times P) \cap (K \times \hat{g}))\big)$.

If $f_i\colon A \to Y_i$ $i = 1, 2$ are sub-analytic mappings, then

$$(f_1, f_2)\colon A \ni x \to (f_1(x), f_2(x)) \in Y_1 \times Y_2$$

is a sub-analytic mapping.

Definition 7. A subset $\Gamma \subset X$ is called a *semi- (sub-) analytic stratum* if Γ is a semi- (sub-) analytic subset of X as well as an analytic submanifold of X.

Definition 8. We define *the dimension of a sub-analytic set* E as $\max(\dim\sigma\colon \sigma \subset E$, σ an analytic submanifold of $X)$.

In [1] there are the proofs of the following properties of the dimension of a sub-analytic set $E \subset X$:

(p. 1.) if $X = U \oplus V$, $\Pi\colon X \to U$ the natural projection then $\dim \Pi(E) \leqslant \dim E$;
(p. 2.) $\dim \bar{E} = \dim E$;

(p. 3.) $\dim E < \dim X \Leftrightarrow E$ is nowhere dense in X;
(p. 4.) $\dim(\bar{E} \setminus E) < \dim E$, if $E \neq \emptyset$;
(p. 5.) $\dim(\bar{E} \setminus \operatorname{int} E) < \dim X$; if $E \subset \Gamma$, Γ and analytic submanifold of X then $\dim(\bar{E} \setminus \operatorname{int} E) < \dim \Gamma$
(p. 6.) $\operatorname{int} \bar{E} \neq \emptyset \Rightarrow \operatorname{int} E \neq \emptyset$.

For the projection of a semi-analytic bounded set we have the following partition [1]:

LEMMA B. *Let $X = U \oplus V$ and Π: $X \to U$ be the natural projection. Then for every semi-analytic bounded subset $A \subset X$ there are semi-analytic stratum Γ_j in a finite number such that $\Pi(A) = \bigcup \Pi(\Gamma_j)$ and Π_{Γ_j}: $\Gamma_j \to U$ is immersion for each j.*

2^0. Let $G_k(X)$ be a Grassmanian manifold (the set of all k-dimensional subspaces of a vector space X). $G_k(X)$ admits the structure of a Nash manifold introduced by an atlas consisting of inverse charts:

$$\varphi_{UV}\colon L(U, V) \ni f \to \hat{f} = \{u + f(u)\colon u \in U\} \in G_k(X),$$

where $\dim U = k$ and $U \oplus V = X$.

Let X be a Euclidean space with an orthonormal base $\{e_1, \ldots, e_n\}$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_1 < \ldots < \alpha_k$ let us denote $U_\alpha = \sum_{t=1}^{K} Re_{\alpha_t}$ then $U_{\alpha'}$, where $\alpha' = (\alpha'_1, \ldots, \alpha'_{n-k})$, $(\alpha'_1, \ldots, \alpha'_{n-k}) = \{1, \ldots, n\} \setminus \{\alpha_1, \ldots, \alpha_k\}$ is the orthogonal complement to U. We shall show that

$$G_k(X) = \bigcup_\alpha G_k^\alpha,$$

where

$$G_k^\alpha = \{\hat{f}\colon f \in K_\alpha\} \quad \text{and} \quad K_\alpha = \left\{f \in L(U_\alpha, U_{\alpha'})\colon |f| < \sqrt{\binom{n}{k}} + 1\right\}.$$

Let us take $W \in G_k(X)$. We should find some $f \in K_\alpha$ such that $W = \hat{f}$. For $w \in \Lambda^k W \subset \Lambda^k X$ (k-th exterior product) such that $|w| = 1$ we have $w = \sum_{\alpha_1 < \ldots < \alpha_k} \lambda_\alpha e_\alpha$, where $e_\alpha = e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_k}$ hence $\sum_\alpha \lambda_\alpha^2 = 1$, so for some α $|\lambda_\alpha| \geqslant \sqrt{\binom{n}{k}}$. Because of $|w \wedge e_{\alpha'}| = |\lambda_\alpha| > 0$, we have $W \oplus U_{\alpha'} = X$, hence there is $f \in L(U_\alpha, U_{\alpha'})$ such that $W = \hat{f}$. Now let us take $x \in W$, $y \in U_{\alpha'}$ such that $|x| = |y| = 1$. $w = x \Lambda w'$, where w' is the exterior product of vectors forming with x an orthonormal base of W. Similarly we can present $e_{\alpha'} = y \wedge e'$. $|w'| = |e'| = 1$, hence from Hadamard's inequality it follows that

$$|w \wedge e_{\alpha'}| = |x \wedge w' \wedge y \wedge e'| \leqslant |x \wedge y|,$$

so we obtain for $u \in U_\alpha$

$$|u|\,|f(u)| \geqslant |u + f(u) \wedge f(u)| \geqslant \sqrt{\binom{n}{k}}\,|u + f(u)|\,|f(u)| \geqslant \sqrt{\binom{n}{k}^{-1}}\,|f(u)|^2$$

and finally $|f(u)| \leqslant \sqrt{\binom{n}{k}}\,|u|$.

Let $\Gamma \subset X$ be an analytic submanifold. We denote by $T_a\Gamma$ the tangent space of Γ at a point a and by $T\Gamma = \{T_a\Gamma\colon a \in \Gamma\} \subset G_k(X)$.

LEMMA 2.1. *Let $A \subset X \oplus Y$ be a semi-analytic bounded set, X, Y finite dimensional vector spaces, $\Pi\colon X \oplus Y \to X$ the natural projection. Then $\Pi(A) = \Pi(\bigcup \Gamma_j)$, where $\Gamma_j \subset A$ are semi-analytic stratum in a finite number such that:*

(1) $\Pi_{\Gamma_j}\colon \Gamma_j \to X$ is an immersion,

(2) $\forall_j \exists_\alpha \Pi(T\Gamma_j) \subset G_k^\alpha \quad (k = \dim \Gamma_j)$.

LEMMA 2.1 follows from Lemma B (p. 1^0) and from the following theorem [6]:

Let $\Gamma \subset X$ be a k-dimensional semi-analytic stratum. Let us consider the mapping $\tau\colon \Gamma \in x \to T_x\Gamma \in G_k(X)$. If G is a semi-algebraic subset of $G_k(X)$ then $\tau^{-1}(G)$ is a semi-analytic subset of X.

From the theorem already mentioned the sets $\Gamma_j \cap \tau^{-1}(\tilde{\Pi}^{-1}(G_k^\alpha))$, where

$$\tilde{\Pi}\colon \tilde{G}_k(X \oplus Y) = \{L \in G_k(X \oplus Y)\colon \dim \Pi(L) = k\} \ni L \to \Pi(L) \in G_k(X),$$

are semi-analytic in $X \oplus Y$ and open (τ is continuous). Hence for the proof of Lemma 2.1 it is sufficient to replace the stratum obtained from Lemma B by the stratum $\Gamma_j \cap \tau^{-1}(\tilde{\Pi}^{-1}(G_k^\alpha))$.

3^0. THEOREM W. *Let X be a Euclidean vector space (finite dimensional). If E is a compact sub-analytic subset of X then E fulfils the Whitney condition.*

Notice that the theorem W is a consequence of the following theorems 3.1–3.4:

THEOREM 3.1. *Every two sub-analytic compact subsets E, $F \subset X$ are regularly separated.*

THEOREM 3.2. *If $E \subset X$ and $F \subset X$ are compact subsets fulfilling the Whitney condition and regularly separated then $E \cup F$ fulfils the Whitney condition as well.*

THEOREM 3.3. *If $E \subset X$ is sub-analytic and bounded then*

$$E = \bigcup W_i \ \textit{(finite union)}$$

where

$$W_i = \{u + \varphi_i(u)\colon u \in \Omega_i\}, \qquad \Omega_i \text{ open in } U_i,$$

$\varphi_i\colon \Omega_i \to V_i$ analytic, sub-analytic, $X = U_i \oplus V_i$ and $\exists K_i \forall u \in \Omega_i |d_u\varphi_i| \leqslant K_i$.

THEOREM 3.4. *If a sub-analytic bounded subset $E \subset X = U \oplus V$ is of the form $\{u + \varphi(u)\colon u \in \Omega\}$, where Ω open in U, $\varphi\colon \Omega \to V$ analytic and $\exists K \forall u \in \Omega |d_u\varphi| \leqslant K$ then its closure fulfils Whitney condition.*

We shall prove theorems 3.1, 3.3, 3.4. The proof of theorem 3.2 may be found in [5].

The proof of theorem 3.1 is based on the two following lemmas.

LEMMA 3.1. *If E is a compact sub-analytic subset of X then $\varphi\colon X \ni x \to \varrho(x, E)$ is a sub-analytic function.*

Proof. The set $T = \{(x, t) \in X \times \boldsymbol{R}\colon \varphi(x) \leqslant t\}$ is sub-analytic because

$$T = \Pi\{(x, y, t) \in X \times E \times \boldsymbol{R}\colon \varrho(x, y) \leqslant t\},$$

where

$$\Pi\colon X \times X \times \boldsymbol{R} \ni (x, y, t) \to (x, t) \in X \times \boldsymbol{R}$$

hence $\phi = \{(x, \varrho(x, E))\colon x \in E\} = \overline{T} \setminus \operatorname{int} T$ is a sub-analytic set as the difference between two sub-analytic sets.

LEMMA 3.2. *If* $\varphi, \psi\colon X \supset K \to \boldsymbol{R}$ *are continuous, sub-analytic, compact as graphs and* $\varphi^{-1}(0) \subset \psi^{-1}(0)$ *then*

$$\exists C, \alpha > 0 \forall_{x \in K} \quad C|\psi(x)|^\alpha \leqslant |\varphi(x)|.$$

Proof. From our assumption $\varphi = \Pi_1(B_1)$ and $\psi = \Pi_2(B_2)$, where for $i = 1, 2$ $\Pi_i\colon X \times \boldsymbol{R} \times Y_i \to X \times \boldsymbol{R}$ and $B_i \subset X \times \boldsymbol{R} \times Y_i$ are semi-analytic compact subsets. Notice that the set

$$Z = \{(\tau, \sigma)\colon \tau = \varphi(x), \sigma = \psi(x) \text{ for some } x \in K\}$$

is sub-analytic as the image of the semi-analytic compact set

$$\{(x, u, v, \tau, \sigma) \in X \times Y_1 \times Y_2 \times \boldsymbol{R}^2\colon (x, \tau, u) \in B_1, (x, \sigma, v) \in B_2\}$$

in the projection $(x, u, v, \tau, \sigma) \to (\tau, \sigma)$. Since every sub-analytic set contained in $\boldsymbol{R}^2$ is semi-analytic [5], Z is semi-analytic. If $\varphi^{-1}(0) = \emptyset$ then the lemma holds trivially because φ reaches the positive minimum on K. Let us assume that $\varphi^{-1}(0) \neq \emptyset$. Z and $\{0\} \times \boldsymbol{R}$ are regularly separated (every two closed semi-analytic sets are regularly separated [5]) and

$$Z \cap (\{0\} \times \boldsymbol{R}) = \{0, 0\} (\text{if } \varphi(x) = 0 \text{ so } \psi(x) = 0)$$

hence there are constants $C, \alpha > 0$ such that for $(\tau, \sigma) \in Z$ we have $|\tau| \geqslant C|\sigma|^\alpha$ which means that $|\varphi(x)| \geqslant C|\psi(x)|^\alpha$ on K.

For the proof of Theorem 3.1 let K be a compact cube containing $E \cap F$. From Lemma 3.1 $\varphi_K\colon K \ni x \to \varrho(x, E)$,

$$\psi_K\colon K \ni x \to \varrho(x, F), \qquad \chi_K\colon K \ni x \to \varrho(x, E \cap F)$$

and (p. 1^0) $\varphi_K + \psi_K$ are sub-analytic continuous compact functions. Because $(\varphi_K + \psi_K)^{-1}(0) = (\chi_K)^{-1}(0)$ we get from Lemma 3.2 constants $C, \alpha > 0$ such that

$$\forall x \in K \, \varrho(x, E) + \varrho(x, F) \geqslant C\varrho(x, E \cap F),$$

which completes the proof.

Now we give the proofs of three lemmas, which we shall need in the proof of Theorem 3.3.

LEMMA 3.3. *Suppose* $X = U \oplus V$, $\Pi\colon X \ni x = u + v \to u \in U$, $E \subset X$ *is a sub-analytic bounded stratum such that* Π_E *is the local diffeomorphism. Then* $v\colon U \ni u \to \#(E \cap (u + V))$ *is a bounded function.*

Proof. We have $\tilde{v}(u)<\infty$ for $u\in U$ because the set $E\cap(u+V)$ is discrete (Π_E local diffeomorphism) and bounded from our assumption and as a sub-analytic set has only a finite number of components.

If $v(u_0)\geqslant k$, then in some neighbourhood U of u_0 there are at least k different sections Π_E in U, hence $v(u)\geqslant k$ for $u\in U$, which shows that $\{u\in U\colon v(u)\geqslant k\}$ is open for $k=0,1,\dots$ This enables us to see that v is a lower semi-continuous function (v is a natural valued function). The set $\Lambda=\Pi(\bar{E}\setminus E)$ is closed, nowhere dense in U (this is a consequence of p. 1, 3, 4 p. 1^0) hence its complement $U\setminus\Lambda$ is open and dense. Because of the lower semi-continuity of v it is sufficient to show that v is bounded in $U\setminus\Lambda$. Let us take an open cube G containing $\Pi(\bar{E})$. We have $v(u)=0$ for $u\in U\setminus G$. $G\setminus\Lambda$ as a sub-analytic set has only a finite number of connected components. We shall show that v is constant in each of them, which completes the proof.

The mapping $\Pi_{E_0}\colon E_0\to G\setminus\Lambda$, where $E_0=(\Pi_E)^{-1}(G\setminus\Lambda)$, is a local homeomorphism, proper (if $K\subset G\setminus\Lambda$, K compact, then $\Pi_{E_0}^{-1}(K)=\Pi^{-1}(K)\cap E=\Pi^{-1}(K)\cap\bar{E}$ is compact) hence Π_{E_0} is a finite covering, therefore v is constant in each connected component of $G\setminus\Lambda$.

LEMMA 3.4. *Let* $\Gamma\subset X\oplus Y$ *be a semi-analytic bounded stratum,* $\dim\Gamma=k$, $\Pi\colon X\oplus Y\to X$ *the natural projection and* Π_Γ *an immersion. Assume that there exists a subspace* $V\subset X$ *such that for each* $z\in\Gamma$ $V\oplus\Pi(T_Z\Gamma)=X$. *Then* $\Pi(\Gamma)=E'\cup E''$, *where* E', E'' *are sub-analytic,* E' *an analytic submanifold and* $\dim E''<\dim E'=k$.

Proof. Let us take U such that $X=U\oplus V$ and consider the natural projection $d\colon X\to U$. V is the common supplement for $\Pi(T_Z\Gamma)$ hence $(p\circ\Pi)_\Gamma$ is a local diffeomorphism. From Lemma 3.3 it follows that function

$$\tilde{v}(u)=\#(\Gamma\cap(u+V+Y))$$

is bounded therefore

$$v\colon X\ni x\to\#\,\Gamma\cap(x+Y)$$

is bounded also (because $\#\Gamma\cap(x+Y)\leqslant\#\Gamma\cap\big(p(x)+V+Y\big)$). Taking

$$\Lambda_r=\{(z_1,\dots,z_r)\in(X\oplus Y)^\Gamma\colon\ \Pi(z_i)=\Pi(z_j),\ z_i\neq z_j\ \text{for}\ i\neq j\}$$

and

$$\Pi_r\colon\ (X\oplus Y)^r\ni(z_1,\dots,z_r)\to z_r\in X\oplus Y$$

we see that subsets

$$B_r=\{x\in X\colon\ v(x)\geqslant\Gamma\}=(\Pi\circ\Pi_\Gamma)(\Gamma^r\cap\Lambda_r)\qquad\text{are sub-analytic}$$

hence

$$A_r=\{x\in X\colon\ \imath(x)=\Gamma\}=B_r\setminus B_{r+1}\qquad\text{are sub-analytic}.$$

Let us now form the sets

$$\tilde{\Gamma}_r=\{x\in\Gamma\colon\ \#(x+Y)\cap\Gamma=\Gamma\quad\text{and}\quad(x+Y)\cap(\bar{\Gamma}\setminus\Gamma)=\emptyset\}$$

$\tilde{\Gamma}_r$ are sub-analytic sets because $\tilde{\Gamma}_r=\big((A_r\setminus\Pi(\bar{\Gamma}\setminus\Gamma))+Y\big)\cap\Gamma$.

Define $\Gamma_r=\operatorname{int}\tilde{\Gamma}_r$. Γ_r are sub-analytic sets (their complements in Γ are $(\overline{\Gamma\setminus\tilde{\Gamma}_r})\cap\Gamma$). We shall show the following observation:

(*) if $\Gamma_r\neq\emptyset$ then $\Pi(\Gamma_r)$ is an analytic submanifold and $\dim\Pi(\Gamma_r)=k$.

For the proof we take $x_0 \in \Pi(\Gamma_r)$. $\Pi^{-1}(x_0) \cap \Gamma = \{z_1^0, \dots, z_r^0\} z_i^0 \neq z_j^0$ when $i \neq j$ and $\Pi^{-1}(x_0) \cap (\bar{\Gamma}\setminus\Gamma) = \emptyset$. For each z_i we choose such a neighbourhood V_i that $\Pi_{\Gamma\cap V_i}$ is an injection and $\Pi(\Gamma \cap V_i)$ is an analytic submanifold, moreover $V_i \cap V_j = \emptyset$ for $i \neq j$. Γ_r is open in Γ so we may take V_1 such that $V_1 \cap \Gamma \subset \Gamma_r$. Evidently for the proof of (*) it is sufficient to show that the germs at x_0 of the sets $\Pi(V_1 \cap \Gamma)$ and $\Pi(\Gamma_r)$ are the same.

Of course we have $\Pi(V_1 \cap \Gamma)_{x_0} \subset \Pi(\Gamma_r)_{x_0}$. Assume $\Pi(V_1 \cap \Gamma)_{x_0} \neq \Pi(\Gamma_r)_{x_0}$. This implies the existence of a sequence $x_n \to x_0$ such that

$$\forall n\, x_n \in \Pi(\Gamma_r)\setminus\Pi(V_1 \cap \Gamma) \tag{**}$$

We have $\Pi^{-1}(x_n) \cap \Gamma = \{z_1^{(n)}, \dots, z_r^{(n)}\}$, $z_i^{(n)} \neq z_j^{(n)}$ for $i \neq j$. Because Γ is bounded we may assume that the sequences $z_i^{(n)}$ are convergent $z_i^{(n)} \to z_i$ for $i = 1, \dots, r$. Notice that for each i

$$\Pi(z_i) = x_0\big(\Pi(z_i^{(n)}) = x_n, \quad \Pi(z_i^{(n)}) \to \Pi(z_i), x_n \to x_0\big),$$

hence $\{z_1, \dots, z_r\} \subset \{z_1^0, \dots, z_r^0\}$. Moreover we have $z_i \neq z_j$ for $i \neq j$ (otherwise supposing $z_i^{(n)} \to z_i^0$ and $z_j^{(n)} \to z_i^0$, $i \neq j$, $z_i^{(n)}$, $z_j^{(n)}$ would be in $V_i \cap \Gamma$ for sufficiently large n and since $\Pi(z_i^{(n)}) = \Pi(z_j^{(n)})$ we get a contradiction with injectivity of $\Pi_{\Gamma\cap V_i}$. So we have proved $\{z_1, \dots, z_r\} = \{z_1^0, \dots, z_r^0\}$. In particular for some i we have $z_i = z_1^0$. The sequence $z_i^{(n)} \to z_1^0$ hence $z_i^{(n)} \in V_1 \cap \Gamma$ for sufficiently large n, therefore $x_n \in \Pi(V_1 \cap \Gamma)$ for sufficiently large n_i which contradicts (**).

Now let us define $E_r = \Pi(\Gamma_r)$. By (*) $E' = \bigcup_i (E_i \setminus \bigcup_{j\neq i} \bar{E}_j)$ is a sub-analytic stratum and $\dim E = k$. For the proof of Lemma 3.4 it suffices to show that $\dim \Pi(\Gamma)\setminus E' < k$,

$$\begin{aligned}\Pi(\Gamma) &= (\Pi(\Gamma)\setminus\Pi(\bigcup \Gamma_i)) \cup \Pi(\bigcup \Gamma_i) \\ &= (\Pi(\Gamma)\setminus\Pi(\bigcup \tilde{\Gamma}_i)) \cup (\Pi(\bigcup \tilde{\Gamma}_i)\setminus\Pi(\bigcup \Gamma_i)) \cup \Pi(\bigcup \Gamma_i)\end{aligned}$$

$$\Pi(\Gamma)\setminus\Pi(\bigcup \tilde{\Gamma}_i) \subset \Pi(\Gamma\setminus\bigcup \tilde{\Gamma}_i) \subset \Pi(\bar{\Gamma}\setminus\Gamma)$$

hence (pp. 1, 4 p. 1^0)

$$\dim \Pi(\Gamma)\setminus\Pi(\bigcup \tilde{\Gamma}_i) < k \tag{1}$$

$$\Pi(\bigcup \tilde{\Gamma}_i)\setminus\Pi(\bigcup \Gamma_i) \subset (\bigcup \tilde{\Gamma}_i \setminus \bigcup \Gamma_i) = (\bigcup \tilde{\Gamma}_i \setminus \mathrm{int}_\Gamma \tilde{\Gamma}_i)$$

hence (pp. 1, 5 p. 1^0)

$$\dim(\Pi(\bigcup \tilde{\Gamma}_i)\setminus\Pi(\bigcup \Gamma_i)) < k \tag{2}$$

$$\Pi(\bigcup \Gamma_i) = \bigcup_i (E_i \setminus \bigcup_{j\neq i} \bar{E}_j) \cup \bigcup_i (E_i \cap (\bigcup_{j\neq i} \bar{E}_j)) = E' \cup F,$$

where

$$F = \bigcup_i (E_i \cap (\bigcup_{j\neq i} \bar{E}_j))$$

since $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\bigcup_i (E_i \cap (\bigcup_{j\neq i} \bar{E}_j)) = \bigcup_i (E_i \cap \bigcup_{j\neq i} (\bar{E}_j \setminus E_j)).$$

Because $\forall i \ \dim(\bar{E}_i \setminus E_i) < k$ (p. 4 p. 1^0) we get

$$\dim F < k, \tag{3}$$

Finally from (1), (2), (3) we conclude that $\dim(\Pi(\Gamma) \setminus E') < k$.

LEMMA 3.5. *Assume* $X = U \oplus V$. *Let* $p\colon X \to U$ *be the natural projection and* $E \subset X$ *a submanifold such that* $p_E\colon E \to U$ *is a local homeomorphism. If* φ *is a section of* p_E *over some neighbourhood* U *of* u_0 ($\varphi\colon U \to V, p_E \circ (id + \varphi) = id_U$) *continuous in* u_0, *then the germs of the sets* E *and* $\hat{\varphi}$ *at* $u_0 + \varphi(u_0)$ *are the same.*

Proof. From our assumption $\hat{\varphi} \subset E$. Denote $x_0 = u_0 + \varphi(u_0)$ and suppose $\hat{\varphi}_{x_0} \neq E_{x_0}$. Then we may construct a sequence $x_n \to x_0$ such that $\forall n \ x_n \in E \setminus \hat{\varphi}$

$$p(x_n) = u_n, \text{ where } x_n = u_n + v_n, \quad p(x_n) \to p(x_0) = u_0$$

and since φ is continuous in u_0 $\varphi(u_n) \to \varphi(u_0)$.
Choosing a neighbourhood G of x_0 such that $p_{G \cap E}$ is an injection we see that for sufficiently large n $x_n = u_n + \varphi(u_n)$ which contradicts (*), therefore $\hat{\varphi}_{x_0} = E_{x_0}$.

Proof of Theorem 3.3 goes by induction on the dimension of a sub-analytic set.

The theorem is evidently true for $\dim E = 0$. Inductively, suppose $\dim E = k$ and all $k-1$ dimensional sub-analytic sets have the required partition. If $k = \dim X$, then $\operatorname{int} E \neq \emptyset$ (pp. 3, 6 1^0) and $\dim(E \setminus \operatorname{int} E) < k$ (p. 4 1^0). We may use induction hypothesis to $E \setminus \operatorname{int} E$ and $\operatorname{int} E$ is a graph of the constant function $\operatorname{int} E \ni x \to 0$.

Therefore we may assume $\dim E < \dim X$. From the definition of a sub-analytic set there exists a semi-analytic bounded set $A \subset X \oplus Y$ such that $E = \Pi(A)$, $\Pi\colon X \oplus Y \to X$.

Applying Lemma 2.1 we obtain

$$\Pi(A) = \Pi\left(\bigcup \Gamma_j\right) \text{ (finite union)},$$

where Γ_j are semi-analytic stratum for which the following conditions hold:

(1) $\forall j \ \Pi_{\Gamma_j}\colon \Gamma_j \to X$ is an immersion,

(2) $\forall j$ such that $\dim \Gamma_j = k \ \exists \alpha$ such that $\Pi(T\Gamma_j) \subset G_k^\alpha$. Since for each of the sets $\Pi(\Gamma_j)$ such that $\dim \Gamma_j < k$ we may use the induction hypothesis, it is sufficient to show that the required decomposition on graphs has every sub-analytic set of the form $\Pi(\Gamma)$ where Γ is a semi-analytic stratum, $\dim \Gamma = k$, $rk \Pi_\Gamma = k$ and $\Pi(T_x \Gamma) \subset G_k^\alpha$ for some α (*).

It follows from (*), Lemma 3.4 and the induction hypothesis that for the proof of our theorem it suffices to construct the required partition for every sub-analytic stratum $E \subset X$ such that $\forall x \in E \ T_x E \in G_k^\alpha$.

Since $\forall x \in E \ T_x E \in G_k^\alpha$ we see that $V = U_{\alpha'}$ is the common complement for $T_x E$, hence p_E is a local diffeomorphism ($p\colon X \to U$ is the natural projection, $U = U_\alpha$ in the definition of G_k^α $U_\alpha \oplus U_{\alpha'} = X$).

From Lemma 3.3 we get the existence of finite

$$s = \max_{u \in U} \#\{(u + V) \cap E\}.$$

Taking

$$A_r = \{(x_1, \ldots, x_r) \in X^r\colon p(x_i) = p(x_j), x_i \neq x_j \text{ for } i \neq j\}$$

and

$$\Pi_r\colon X^r \ni (x_1, \ldots, x_r) \to x_r \in X$$

we see that the sets

$$B_r = \{u \in U\colon (u+V) \cap E \geqslant r\} = (p \circ \Pi_r)(E^r \cap \Lambda_r)$$

are sub-analytic, hence the sets

$$\Omega_r = B_r \setminus B_{r+1} = \{u \in U\colon \#(u+V) \cap E = r\}, \qquad r = 1, \ldots, s$$

are also sub-analytic.

We have $\Omega = p(E) = \bigcup_{r=1}^{s} \Omega_r$. Let us denote $\tilde{E}_r = (\Omega_r + V) \cap E$ and $E_r = \operatorname{int}_E \tilde{E}_r$. We shall construct the required partition for the sets $E_r \neq \emptyset$. E_r is a k-dimensional sub-analytic stratum as an open subset of E and p_{E_r} is a local diffeomorphism, hence (Lemma 3.5) if φ is some section of p_E defined in a neighbourhood of u_0 and continuous in u_0 then the germs at $u_0 + \varphi(u_0)$ of the sets E_r and $\hat{\varphi}$ are the same. First we shall show that $E_r = \bigcup_{j=1}^{r} \tilde{F}^r_j$, where $\tilde{F}^r_j$ are sub-analytic sets and graphs of function $\tilde{\varphi}^r_j\colon p(E_r) \to V$.

Let $\{e_1, \ldots, e_n\}$ be a base of X such that $\{e_1, \ldots, e_k\}$ form a base of U. For each $x \in X$ we have $x = \sum_{i=1}^{n} \lambda_i(x) \cdot e_i$.

Let us define for $x_1, x_2 \in X$

$$x_1 R x_2 \Leftrightarrow (1)\ p(x_1) = p(x_2)(\forall_{i \leqslant k} \lambda_i(x_1) = \lambda_i(x_2)) \quad \text{and}$$
$$(2)\ \exists m \forall p < m\, \lambda_{k+p}(x_1) = \lambda_{k+p}(x_2) \quad \text{and} \quad \lambda_{k+m}(x_1) < \lambda_{k+m}(x_2).$$

The relation R orders each fibre $p^{-1}(u)$, where $u \in U$.
The set

$$P_r = \{(x_1, \ldots, x_r)\colon x_i \in E_i\ i = 1, \ldots, r,\ x_1 R x_2 R \ldots R x_r\}$$

is sub-analytic as an intersection of the Cartesian product of sub-analytic sets with a semi-algebraic set.

Let $p^r_j\colon X^r \ni (x_1, \ldots, x_r) \to x_j \in X$ be the projection. The set $\tilde{F}^r_j = p^r_j(P_r)$ is sub-analytic as the projection of P_r. $x \in \tilde{F}^r_j$ means exactly that x is the j-th point of the fibre $p^{-1}_{E_r}(p(x))$ considering the order R.

$$E_r = \bigcup_{j=1}^{r} \tilde{F}^r_j \quad \text{and } \tilde{F}^r_j = \{u + \tilde{\varphi}^r_j(u)\colon u \in p(E_r)\},$$

where: $\tilde{\varphi}_r\colon p(E_r) \to V$ is a section of p_{E_r}.

S^r_j, the set consisting of all discontinuity points of $\tilde{\varphi}^r_j$, is a sub-analytic set as equal to $p(\bar{\hat{\varphi}}^r_j \setminus \hat{\tilde{\varphi}}^r_j) \cap p(E_r)$ and $\dim S^r_j < k$ (pp. 1,4 1^0).

Let us put $F^r_j = \tilde{F}^r_j \setminus ((S^r_j + V) \cap E)$.

We apply the induction hypothesis to the sub-analytic set

$$(\overline{S^r_j} + V) \cap E_r (\dim (\overline{S^r_j} + V) \cap E_r < k),$$