

Malgrange-Zerner theorem in infinite dimensional case

by Ludwik M. DRUŻKOWSKI

Introduction. At first we quote a generalization of the Malgrange-Zerner Theorem ([3]). Let $P := \{w \in \mathbb{C}: 0 \leq \operatorname{Im} w < r\} = \mathbb{R} + i[0, r)$ be the strip in the complex plane \mathbb{C} , let $X := (\mathbb{R} \times P) \cup (P \times \mathbb{R})$ be the subset of \mathbb{C}^2 and let $f: X \rightarrow \mathbb{C}$ be a separately holomorphic function (i.e., for each fixed $x \in \mathbb{R}$ the function $f(x, \cdot)$ is holomorphic in the interior of the strip P and continuous in P , and the same is required for the second argument). If in addition f is locally bounded and f restricted to \mathbb{R}^2 is continuous, then f has the unique continuous extension \hat{f} to the convex hull of X , so that \hat{f} is holomorphic in the interior of $\operatorname{conv} X$ and

$$(*) \quad \sup \{|\hat{f}(z)|: z \in \operatorname{conv} X\} = \sup \{|f(z)|: z \in X\}.$$

It is natural to ask: will the above theorem be true when we take an arbitrary real Hausdorff topological vector space (t.v.s.) E in the place of \mathbb{R} and its complexification ([1], [5]) \tilde{E} as \mathbb{C} ?

More exactly, let F be a sequentially complete locally convex space (s-c.l.c.s.) over \mathbb{C} and $f: U \rightarrow F$, where U is an open subset of a t.v.s. E over \mathbb{C} ([1], [5]). We say f is G -holomorphic in U iff for every affine line L in E the function f restricted to $U \cap L$ is holomorphic; f is holomorphic if f G -holomorphic and continuous [2]. We will prove the following

THEOREM. Let F be a s-c.l.c.s. over \mathbb{C} let E be a real t.v.s., let \tilde{E} denote the complexification of E (we identify $E = E + i0$) and let further 1 be a nontrivial continuous linear functional on *E . Putting

$$A := \{x \in E: 0 \leq 1(x) < 1\}, P := E + iA, T := (E \times P) \cup (P \times E) \subset \tilde{E} \times \tilde{E} = \overline{E \times E}$$

we define a function $f: T \rightarrow F$ that has the following properties

- (1) f is a separately holomorphic function in T , i.e., for each fixed $x \in E$ $f(x, \cdot)$ is holomorphic in $\operatorname{int} P$ and continuous in P , and for each fixed $y \in E$ $f(\cdot, y)$ is holomorphic in $\operatorname{int} P$ and continuous in P ;
- (2) f restricted to $E \times E$ is continuous;
- (3) f is locally bounded in T .

Then f has the unique extension $\hat{f}: W := \operatorname{conv} T \rightarrow F$ such that \hat{f} is holomorphic in $\operatorname{int} W$ and for every finite dimensional subspace S of $\tilde{E} \times \tilde{E}$ the function \hat{f} is continuous in $W \cap S$ and

for every continuous seminorm q in F we have

$$\sup\{q \circ f(x) : x \in W\} = \sup\{q \circ f(x) : x \in T\}.$$

Proof of the Theorem.

(i) $F = C$. Let us put $S := \tilde{E}_s \times \tilde{E}_s$, where E_s is a finite dimensional subspace of E . Since f restricted to $T \cap S$ fulfills the assumptions of the Malgrange-Zerner Theorem (the generalized version in C^n , [3]), so $f|_{T \cap S}$ has the unique extension \hat{f}_S into $\text{conv}(W \cap S)$. We obtain the extension \hat{f} of f to the set $\text{conv}T$ as glueing of family of finite dimensional extensions \hat{f}_S that are identical along their common domains. This extension is evidently G -holomorphic in $\text{int}W$, for every finite dimensional subspace S of the mentioned type, \hat{f} restricted to $W \cap S$ is continuous and the equality (*) holds in $W \cap S$. We see that the whole problem is to check continuity of \hat{f} in $\text{int}W$ that is equivalent to a local boundedness of \hat{f} ([2]). We need the following

LEMMA. Let

$$Y := \{[-p, p] \times ([-p, p] + i[0, r])\} \cup \{([-p, p] + i[0, r]) \times [-p, p]\},$$

$$r > 1, p \geq \frac{1+r}{2} \sqrt{\frac{2r}{r-1}},$$

be a subset of C^2 . There exists a set $D \subset C^2$ such that $\text{int}D$ contains

$$\text{conv}\left\{\left(0 \times \left(0 + i\left(0, \frac{r+1}{2}\right)\right)\right) \cup \left(\left(0 + i\left(0, \frac{r+1}{2}\right)\right) \times 0\right)\right\}$$

and for every function f that is continuous and separately holomorphic in Y there exists the unique extension \hat{f} continuous in D and holomorphic in $\text{int}D$ and

$$\sup\{|\hat{f}(z)| : z \in D\} \leq \sup\{|f(z)| : z \in Y\}.$$

(Proof of Lemma can be obtained by M-Z Theorem ([3], [4]) and by suitable calculations).

Let us take $w = (a' + ia, b' + ib) \in \text{int}W \subset \tilde{E} \times \tilde{E} = \overline{E \times E} = (E \times E) + i(E \times E)$. By dint of the form of T ($T = (E \times E) + i\{(0 \times A) \cup (A \times 0)\}$) we have that

$$(a, b) \in \text{int} \text{conv}\{(A \times 0) \cup (0 \times A)\} \subset E \times E$$

and by convexity of A we get $(a, b) = m(c, 0) + (1-m)(0, d)$, $0 < m < 1$, $c, d \in \text{int}A \subset E$. We take Y the same as in Lemma (with r close to 1 such that $rc, rd \in A$) and we put

$$Z := \{(a', b') + s(c, 0) + t(0, d) \in \tilde{E} \times \tilde{E} : (s, t) \in Y\}.$$

Therefore Z is a compact subset of T (T is endowed with the relative topology from $\tilde{E} \times \tilde{E}$). Due to local boundedness of f in T there exists $M > 0$ such that for each $x \in Z$ there exists an open balanced set U_x in E such that f is bounded by M in the set $x + B_x \subset T$, where B_x according to x has the form $\{(U_x \times U_x) + i(U_x \times U_x)\} \cap T$. Since Z is compact so there

exists an open balanced V in E such that

$$(a', b') + i(rc, 0) + \{(V \times V) + i(V \times 0)\} \subset T,$$

$$(a', b') + i(0, rd) + \{(V \times V) + i(0 \times V)\} \subset T$$

and

$$z + T \cap \{(V \times V) + i(V \times V)\} \subset U := \bigcup_{x \in Z} (x + B_x), \quad z \in Z.$$

In particular

$$Z_V := \{(a', b') + (u', v') + s(c+u, 0) + t(0, d+v) : (s, t) \in Y; u', v', u, v \in V\}$$

is contained in U , so f is bounded by M in Z_V . For every

$$z = (u' v') + i(u, v) \in B := (V \times V) + i(V \times V)$$

we define

$$g_z(s, t) := \hat{f}[(a', b') + (u', v') + s(c+u, 0) + t(0, d+v)],$$

where $(s, t) \in \text{conv } Y \subset \mathbb{C}^2$; in particular $f(w) = g_0(im, i(1-m))$. Let us take $z = (u', v') + i(u, v) \in kB$, $k := \min(m, 1-m)$ and let us put

$$z' := (u', v') + i\left(\frac{u}{m}, \frac{v}{m-1}\right) \in B,$$

by Lemma we have

$$|\hat{f}(w+z)| = |g_z(im, i(1-m))| \leq \sup\{|g_z(s, t)| : (s, t) \in Y\} \\ \leq \sup\{|f(x)| : x \in Z_V\} \leq M$$

and therefore \hat{f} is continuous.

(ii) F is an arbitrary s-c.l.c.s. over \mathbb{C} . Keeping on the notation of the preceding case we have the inequality:

$$|u \circ \hat{f}(w+z)| \leq M_q := \sup\{q \circ f(x) : x \in U\}$$

for every continuous linear form u in F such that $|u(y)| \leq q(y)$ for a fixed continuous seminorm q in F . So by Hahn-Banach Theorem we obtain

$$q \circ \hat{f}(w+z) \leq M_q \quad \text{for } z \in kB.$$

The proof of Theorem is complete.

Problem. Is \hat{f} continuous in W in this case? More generally, if f is holomorphic in a domain of holomorphy U , continuous in the whole boundary of U and for every finite dimensional subspace S of E , f is continuous in $\bar{U} \cap S$ is then f continuous in \bar{U} ?

References

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INSTITUTE OF MATHEMATICS
JAGELLONIAN UNIVERSITY
KRAKÓW (POLAND)