

On the invariance of the L -regularity under holomorphic mappings

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1. Introduction. Let E be a compact set in C^n and let $P(C^n)$ denote the space of all polynomials of n complex variables. The Siciak extremal function of the set E is defined as follows (see [8] and [9]):

$$\Phi(z, E) = \sup \{ |p(z)|^{1/\deg p} : p \in P(C^n), \|p\|_E \leq 1, \deg p \geq 1 \},$$

for $z \in C^n$, where $\|p\|_E = \sup \{ |p(w)| : w \in E \}$. If E is a compact subset of the complex plane C , then the function $\log \Phi(z, E)$ is known to be equal to the Green function of the unbounded component of $C \setminus E$ with pole at infinity. It is well known that the Green function plays an important role in the theory of interpolation and approximation of holomorphic functions of one variable by polynomials. The Siciak extremal function plays a very similar role in the theory of functions of several complex variables (see [8]).

The assumption that the function $\Phi(z, E)$ is continuous at some points of C^n is often necessary in applications of the method of extremal functions. In the case of one complex variable the question of the continuity of $\Phi(z, E)$ has been well explored. The analogous problem in higher dimensions is much more involved. Some criteria for the continuity of $\Phi(z, E)$ may be found in the literature (see [6], [7], [8], [9]). Additional information may be derived from the main results of this paper (Theorems: 1.1, 1.2, 1.3, 1.4).

We now recall some useful definitions. Let E be a compact set in C^n . We say that E is L -regular at a point $a \in E$ if the function $\Phi(z, E)$ is continuous at a . If $\Phi(z, E)$ is continuous in C^n we say that E is L -regular. By a result of Zaharjuta (see [9]) in order that E be L -regular it suffices that E be L -regular at every point $a \in E$. The set E is called *locally L -regular at a point $a \in E$* if for every $r > 0$ the extremal function $\Phi(z, E \cap \bar{B}(a, r))$ is continuous at a , where $\bar{B}(a, r)$ is the closed Euclidean ball in C^n with centre at a and radius r .

We may now state our results.

THEOREM 1.1. *Let f be a holomorphic mapping defined in an open subset of C^n , with values in C^n . Let E be a compact set in C^n such that the inverse image of the polynomially convex hull of E under f is compact. If $a \in f^{-1}(E)$ and E is L -regular at the point $f(a)$, then $f^{-1}(E)$ is L -regular at the point a .*

THEOREM 1.2. *Let f be a holomorphic mapping defined in an open subset of C^n , with values in C^n . Let E be a compact set in C^n such that the inverse image of E under f is compact.*

If $a \in f^{-1}(E)$ and E is locally L -regular at the point $f(a)$, then $f^{-1}(E)$ is locally L -regular at the point a .

THEOREM 1.3. Let E be a compact set in \mathbf{C}^n . Let f be a holomorphic mapping defined in a neighbourhood of the polynomially convex hull of E , with values in \mathbf{C}^m ($m \leq n$). If E is L -regular at a point $a \in E$ and f is open in a neighbourhood of a , then $f(E)$ is L -regular at the point $f(a)$.

If $p \in P(\mathbf{C}^n)$ and $\deg p \geq 1$, then \hat{p} denote a homogeneous polynomial, such that $\deg \hat{p} = \deg p$ and $\deg(p - \hat{p}) < \deg p$.

THEOREM 1.4. Let $f = (f_1, \dots, f_n): \mathbf{C}^n \rightarrow \mathbf{C}^n$ be a polynomial mapping. Put $d = \max\{\deg f_j: 1 \leq j \leq n\}$. The following conditions are equivalent:

(1) $\liminf_{|z| \rightarrow +\infty} (|f(z)|/|z|^d) > 0$ and $d \geq 1$;

(2) $\deg f_1 = \dots = \deg f_n = d \geq 1$ and $\hat{f}^{-1}(0) = \{0\}$, where $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$;

(3) The mapping f is proper and for every compact set E in \mathbf{C}^n

$$\Phi(z, f^{-1}(E)) = (\Phi(f(z), E))^{1/d}, \quad z \in \mathbf{C}^n;$$

(4) The mapping f is proper and for every compact set E in \mathbf{C}^n

$$(\Phi(w, E))^{1/d} = \max\{\Phi(z, f^{-1}(E)): z \in f^{-1}(w)\}, \quad w \in \mathbf{C}^n;$$

(5) The mapping f is proper and there is a compact set E in \mathbf{C}^n such that E is not polar and

$$\Phi(z, f^{-1}(E)) = (\Phi(f(z), E))^{1/d}, \quad z \in \mathbf{C}^n;$$

(6) The mapping f is proper and there is a compact set E in \mathbf{C}^n such that E is not polar and

$$(\Phi(w, E))^{1/d} = \max\{\Phi(z, f^{-1}(E)): z \in f^{-1}(w)\}, \quad w \in \mathbf{C}^n.$$

Theorem 1.3 is similar to a recent result of Pleśniak [6]. Assume that E is as in Theorem 1.3 and $F \subset E$. Pleśniak considered a large class $\mathcal{H}(F)$ of mappings including all mappings which are holomorphic in a neighbourhood of E and open in a neighbourhood of F . Theorem 1.3 was proved in [6] under the assumption that $f \in \mathcal{H}(E)$, E is polynomially convex and L -regular (see [6], Theorem 3.5) or $f \in \mathcal{H}(\{a\})$ and $E \subset \mathbf{R}^n$ (see [6], Theorem 3.8).

As a consequence of Theorem 1.4 we obtain a generalization of some results of Polya and Fekete on transfinite diameter (logarithmic capacity) of plane sets (see [1]).

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2. Notation. If f is a mapping and a set A is contained in the domain of f , then we denote by $f|A$ the restriction of f to A . If a is a member of the range of f , then we write $f^{-1}(a)$, instead of $f^{-1}(\{a\})$. Let X be a topological space. If A is a subset of X , then \bar{A}