

On the invariance of the L -regularity under holomorphic mappings

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1. Introduction. Let E be a compact set in C^n and let $P(C^n)$ denote the space of all polynomials of n complex variables. The Siciak extremal function of the set E is defined as follows (see [8] and [9]):

$$\Phi(z, E) = \sup \{ |p(z)|^{1/\deg p} : p \in P(C^n), \|p\|_E \leq 1, \deg p \geq 1 \},$$

for $z \in C^n$, where $\|p\|_E = \sup \{ |p(w)| : w \in E \}$. If E is a compact subset of the complex plane C , then the function $\log \Phi(z, E)$ is known to be equal to the Green function of the unbounded component of $C \setminus E$ with pole at infinity. It is well known that the Green function plays an important role in the theory of interpolation and approximation of holomorphic functions of one variable by polynomials. The Siciak extremal function plays a very similar role in the theory of functions of several complex variables (see [8]).

The assumption that the function $\Phi(z, E)$ is continuous at some points of C^n is often necessary in applications of the method of extremal functions. In the case of one complex variable the question of the continuity of $\Phi(z, E)$ has been well explored. The analogous problem in higher dimensions is much more involved. Some criteria for the continuity of $\Phi(z, E)$ may be found in the literature (see [6], [7], [8], [9]). Additional information may be derived from the main results of this paper (Theorems: 1.1, 1.2, 1.3, 1.4).

We now recall some useful definitions. Let E be a compact set in C^n . We say that E is L -regular at a point $a \in E$ if the function $\Phi(z, E)$ is continuous at a . If $\Phi(z, E)$ is continuous in C^n we say that E is L -regular. By a result of Zaharjuta (see [9]) in order that E be L -regular it suffices that E be L -regular at every point $a \in E$. The set E is called *locally L -regular at a point $a \in E$* if for every $r > 0$ the extremal function $\Phi(z, E \cap \bar{B}(a, r))$ is continuous at a , where $\bar{B}(a, r)$ is the closed Euclidean ball in C^n with centre at a and radius r .

We may now state our results.

THEOREM 1.1. *Let f be a holomorphic mapping defined in an open subset of C^n , with values in C^n . Let E be a compact set in C^n such that the inverse image of the polynomially convex hull of E under f is compact. If $a \in f^{-1}(E)$ and E is L -regular at the point $f(a)$, then $f^{-1}(E)$ is L -regular at the point a .*

THEOREM 1.2. *Let f be a holomorphic mapping defined in an open subset of C^n , with values in C^n . Let E be a compact set in C^n such that the inverse image of E under f is compact.*

If $a \in f^{-1}(E)$ and E is locally L -regular at the point $f(a)$, then $f^{-1}(E)$ is locally L -regular at the point a .

THEOREM 1.3. Let E be a compact set in \mathbf{C}^n . Let f be a holomorphic mapping defined in a neighbourhood of the polynomially convex hull of E , with values in \mathbf{C}^m ($m \leq n$). If E is L -regular at a point $a \in E$ and f is open in a neighbourhood of a , then $f(E)$ is L -regular at the point $f(a)$.

If $p \in P(\mathbf{C}^n)$ and $\deg p \geq 1$, then \hat{p} denote a homogeneous polynomial, such that $\deg \hat{p} = \deg p$ and $\deg(p - \hat{p}) < \deg p$.

THEOREM 1.4. Let $f = (f_1, \dots, f_n): \mathbf{C}^n \rightarrow \mathbf{C}^n$ be a polynomial mapping. Put $d = \max\{\deg f_j: 1 \leq j \leq n\}$. The following conditions are equivalent:

(1) $\liminf_{|z| \rightarrow +\infty} (|f(z)|/|z|^d) > 0$ and $d \geq 1$;

(2) $\deg f_1 = \dots = \deg f_n = d \geq 1$ and $\hat{f}^{-1}(0) = \{0\}$, where $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$;

(3) The mapping f is proper and for every compact set E in \mathbf{C}^n

$$\Phi(z, f^{-1}(E)) = (\Phi(f(z), E))^{1/d}, \quad z \in \mathbf{C}^n;$$

(4) The mapping f is proper and for every compact set E in \mathbf{C}^n

$$(\Phi(w, E))^{1/d} = \max\{\Phi(z, f^{-1}(E)): z \in f^{-1}(w)\}, \quad w \in \mathbf{C}^n;$$

(5) The mapping f is proper and there is a compact set E in \mathbf{C}^n such that E is not polar and

$$\Phi(z, f^{-1}(E)) = (\Phi(f(z), E))^{1/d}, \quad z \in \mathbf{C}^n;$$

(6) The mapping f is proper and there is a compact set E in \mathbf{C}^n such that E is not polar and

$$(\Phi(w, E))^{1/d} = \max\{\Phi(z, f^{-1}(E)): z \in f^{-1}(w)\}, \quad w \in \mathbf{C}^n.$$

Theorem 1.3 is similar to a recent result of Pleśniak [6]. Assume that E is as in Theorem 1.3 and $F \subset E$. Pleśniak considered a large class $\mathcal{H}(F)$ of mappings including all mappings which are holomorphic in a neighbourhood of E and open in a neighbourhood of F . Theorem 1.3 was proved in [6] under the assumption that $f \in \mathcal{H}(E)$, E is polynomially convex and L -regular (see [6], Theorem 3.5) or $f \in \mathcal{H}(\{a\})$ and $E \subset \mathbf{R}^n$ (see [6], Theorem 3.8).

As a consequence of Theorem 1.4 we obtain a generalization of some results of Polya and Fekete on transfinite diameter (logarithmic capacity) of plane sets (see [1]).

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2. Notation. If f is a mapping and a set A is contained in the domain of f , then we denote by $f|A$ the restriction of f to A . If a is a member of the range of f , then we write $f^{-1}(a)$, instead of $f^{-1}(\{a\})$. Let X be a topological space. If A is a subset of X , then \bar{A}

and ∂A denote the closure and the boundary of the set A , respectively. Let v be a function from A to $[-\infty, +\infty)$. We put

$$v^*(x) = \limsup_{\substack{y \rightarrow x \\ y \in A}} v(y), \quad x \in \bar{A}$$

and

$$v^+(x) = \max\{0, v(x)\}, \quad x \in A.$$

Let Ω be an open set in \mathbb{C}^n . We denote by $\mathcal{C}(\Omega)$, $O(\Omega)$ and $\text{PSH}(\Omega)$ the space of real-valued continuous functions in Ω , the space of holomorphic functions in Ω and the set of all plurisubharmonic functions in Ω , respectively. If E is a compact subset of Ω and v is a complex valued function defined in E , then we define

$$\begin{aligned} \|v\|_E &= \sup\{|v(z)|: z \in E\}, \\ \hat{E} &= \{z \in \mathbb{C}^n: |p(z)| \leq \|p\|_E, p \in P(\mathbb{C}^n)\}, \\ \hat{E}_{O(\Omega)} &= \{z \in \Omega: |f(z)| \leq \|f\|_E, f \in O(\Omega)\}, \\ \hat{E}_{\text{PSH}(\Omega)} &= \{z \in \Omega: u(z) \leq \sup_E u, u \in \text{PSH}(\Omega)\}. \end{aligned}$$

We denote by $|z|$ the Euclidean norm of $z \in \mathbb{C}^n$. If $a \in \mathbb{C}^n$ and r is a positive number, then $B(a, r)$ is the open r -ball about a .

3. Basic properties of extremal functions. To prove our theorems, we need some results involving extremal functions. For a more detailed discussion of similar results, the reader may consult the literature (see especially [8], [9]). We begin by introducing our terminology.

Given a subset E of \mathbb{C}^n and an open set $G \subset \mathbb{C}^n$ we define for every $z \in G$

$$h(z, E, G) = \sup\{u(z): u \in \text{PSH}(G), u \leq 0 \text{ on } E \cap G, u \leq 1 \text{ on } G\}.$$

We say that $u: \mathbb{C}^n \rightarrow [-\infty, +\infty)$ is a L -function if $u \in \text{PSH}(\mathbb{C}^n)$ and

$$\sup\{u(z) - \log^+ |z|: z \in \mathbb{C}^n\} < +\infty.$$

Denote by L the family of all L -functions. Observe that if $p \in P(\mathbb{C}^n)$ and $\text{deg } p \geq 1$ then $(1/\text{deg } p) \log |p| \in L$.

Let E be any subset of \mathbb{C}^n . We put

$$V(z, E) = \sup\{u(z): u \in L, u \leq 0 \text{ on } E\}.$$

The function $V(z, E)$ is called the L -extremal function associated with E .

THEOREM 3.1 (see [9], Theorem 4.12). *If E is a compact subset of \mathbb{C}^n then*

$$\Phi(z, E) = \exp V(z, E), \quad z \in \mathbb{C}^n.$$

THEOREM 3.2 (see [9], Prop. 6.2). *If E is a compact subset of \mathbb{C}^n then E is L -regular at $a \in E$ if and only if for every bounded neighbourhood G of \hat{E} the function $h(z, E, G)$ is continuous at a .*

We say that a subset E of \mathbb{C}^n is *polar*, if there exists a function $u \in \text{PSH}(\mathbb{C}^n)$ such that $u = -\infty$ on E .

Given a set $E \subset \mathbb{C}^n$ let (see [9]):

$$c(E) = \limsup_{|z| \rightarrow +\infty} (|z| / \exp(-V^*(z, E))).$$

The number $c(E)$ is called the *L-capacity* of E . If E is a compact subset of the complex plane then $c(E)$ is equal to the logarithmic capacity (transfinite diameter) of E .

THEOREM 3.3 (see [9], Cor. 3.9 and Theorem 3.10). *For every subset E of \mathbb{C}^n the following conditions are equivalent:*

- (1) $c(E) = 0$,
- (2) E is polar,
- (3) $V^*(z, E) \notin L$.

Remark 3.4. The following formula holds:

$$\Phi(z, \bar{B}(a, r)) = \max \left\{ 1, \frac{|z-a|}{r} \right\}, \quad z \in \mathbb{C}^n$$

where $a \in \mathbb{C}^n$ and $r > 0$.

4. The L-regularity of images and pre-images of compact sets under holomorphic mappings. In this section we shall prove Theorems 1.1, 1.2 and 1.3.

LEMMA 4.1. *Let Ω and Ω' be open sets in \mathbb{C}^n and let f be a proper holomorphic mapping of Ω onto Ω' . If $u \in \text{PSH}(\Omega)$, then the formula*

$$v(w) = \max \{ u(z) : z \in f^{-1}(w) \}, \quad w \in \Omega'$$

defines a plurisubharmonic function.

Proof. Without loss of generality we may assume that the set Ω' is connected. Denote by A the zero locus of the Jacobian of f . The set A is an analytic subvariety in Ω . Thus $f(A)$ is an analytic subvariety in Ω' , by the proper mapping theorem (see for instance [3]).

First assume that $u \in \mathcal{C}(\Omega) \cap \text{PSH}(\Omega)$. The proper mapping f is a ramified analytic cover. Hence f is open (see [3]) and closed. Let a and b be real numbers ($a < b$). From the definition of the function v it follows that

$$v^{-1}((a, b)) = f(u^{-1}((a, +\infty))) \setminus f(u^{-1}([b, +\infty))).$$

This implies that the function v is continuous. Clearly $v \in \text{PSH}(\Omega' \setminus f(A))$. The function $(v|_{\Omega' \setminus f(A)})$ extends to a plurisubharmonic function in Ω' , by the Grauert–Remmert theorem (see [2]). The extension of $(v|_{\Omega' \setminus f(A)})$ must be equal to $(v|_{\Omega' \setminus f(A)})^*$. But v is continuous, so that $v = (v|_{\Omega' \setminus f(A)})^* \in \text{PSH}(\Omega')$.

Let us now consider the general case. Assume that $u \in \text{PSH}(\Omega)$. Let w_0 belong to Ω' . Choose $r > 0$ so small that $B = B(w_0, r) \subset \bar{B} \subset \Omega'$. Define: $u_0 = u|_{f^{-1}(B)}$. The set $f^{-1}(B)$

is relatively compact in Ω . Therefore there exists a decreasing sequence $\{u_1, u_2, \dots\} \subset \text{PSH}(f^{-1}(B)) \cap \mathcal{C}(f^{-1}(B))$ such that $u_0 = \lim_{n \rightarrow +\infty} u_n$. Define $v_n: B \rightarrow \mathbb{R}$ by letting

$$v_n(w) = \max\{u_n(z): z \in f^{-1}(w)\}, \quad w \in B, \quad n = 0, 1, 2, \dots$$

From the first part of the proof it follows that $v_n \in \text{PSH}(B)$ for $n \neq 0$. The sequence $\{v_1, v_2, \dots\}$ is decreasing and $v_0 = \lim_{n \rightarrow +\infty} v_n$. Therefore $v|_B = v_0 \in \text{PSH}(B)$.

LEMMA 4.2. *Let Ω and Ω' be open sets in \mathbb{C}^n and let f be a proper holomorphic mapping of Ω onto Ω' . If $E \subset \Omega'$, then*

$$h(z, f^{-1}(E), \Omega) = h(f(z), E, \Omega'), \quad \text{for } z \in \Omega.$$

Proof. It is clear that $h(z, f^{-1}(E), \Omega) \geq h(f(z), E, \Omega')$. Let us suppose that $u \in \text{PSH}(\Omega)$, $u \leq 0$ on $f^{-1}(E)$ and $u \leq 1$ on Ω . It follows from Lemma 4.1 that the formula

$$v(w) = \max\{u(z): z \in f^{-1}(w)\}, \quad \text{for } w \in \Omega',$$

defines a plurisubharmonic function in Ω' , such that $v \leq 0$ on E and $v \leq 1$ on Ω' . Hence

$$u(z) \leq v(f(z)) \leq h(f(z), E, \Omega'), \quad z \in \Omega.$$

Therefore $h(z, f^{-1}(E), \Omega) \leq h(f(z), E, \Omega')$.

LEMMA 4.3. *Let $f: X \rightarrow Y$ be a continuous mapping of a locally compact space X into a Hausdorff space Y . Let E be a compact subset of Y such that $f^{-1}(E)$ is compact. Then there is an open neighbourhood U of $f^{-1}(E)$ and there is an open set V in Y such that $f(U) \subset V$ and $f: U \rightarrow V$ is proper.*

Proof. Let B be an open neighbourhood of $f^{-1}(E)$ such that \bar{B} is compact. Define $V = Y \setminus f(\partial B)$, $U = f^{-1}(V) \cap B$. Let K be a compact set in V . Put $L = f^{-1}(K) \cap U$. Clearly $\bar{L} \subset \bar{B}$. It is enough to show that $\bar{L} \subset U$. The set $f^{-1}(K)$ is closed, therefore $\bar{L} \subset f^{-1}(K) \subset f^{-1}(V)$. Observe that $\bar{L} \cap \partial B = \emptyset$ because $f(\bar{L} \cap \partial B) \subset f(\bar{L}) \cap f(\partial B) \subset V \cap f(\partial B) = \emptyset$. Hence $\bar{L} \subset B$.

LEMMA 4.4. *Let E be a compact set in \mathbb{C}^n and let f be a holomorphic mapping defined in an open subset of \mathbb{C}^n , with values in \mathbb{C}^n . If $f^{-1}(\hat{E})$ is compact, then*

$$f^{-1}(\hat{E}) \subset \widehat{f^{-1}(E)} \subset \widehat{f^{-1}(\hat{E})}.$$

Proof. Obviously, the set $\widehat{f^{-1}(E)}$ is contained in $\widehat{f^{-1}(\hat{E})}$. It suffices to show that the first inclusion is true. A proper holomorphic mapping between two open sets in \mathbb{C}^n is open (see [3]). Therefore, by virtue of Lemma 4.3, there exist open sets $\Omega, \Omega' \subset \mathbb{C}^n$ such that $f: \Omega \rightarrow \Omega'$ is proper, $f(\Omega) = \Omega'$, $f^{-1}(\hat{E}) \subset \Omega$ and $\partial\Omega' \cap \hat{E} = \emptyset$. Take polynomials P_1, \dots, P_m (see [4], Lemma 2.7.4) such that

$$\hat{E} \subset D_0 = \{z \in \mathbb{C}^n: |P_j(z)| < 1, j = 1, \dots, m\} \subset \mathbb{C}^n \setminus \partial\Omega'.$$

Define $D_1 = D_0 \cap \Omega'$ and $D_2 = D_0 \cap (\mathbb{C}^n \setminus \bar{\Omega}')$. The open sets D_0, D_1, D_2 are Runge domains (see [4], Theorem 2.7.3 (iv)). Therefore for every compact set $K \subset D_i$ we have

(see [4], Theorems: 2.7.3, 4.2.8, 4.3.4):

$$\hat{K} = \hat{K}_{O(D_i)} = \hat{K}_{\text{PSH}(D_i)}, \quad i = 0, 1, 2.$$

We know that

$$\begin{aligned} f^{-1}(\hat{E}) &= f^{-1}(\hat{E}_{\text{PSH}(D_0)}) = f^{-1}((\widehat{E \cap D_1})_{\text{PSH}(D_1)} \cup (\widehat{E \cap D_2})_{\text{PSH}(D_2)}) = \\ &= f^{-1}((\widehat{E \cap D_1})_{\text{PSH}(D_1)}). \end{aligned}$$

Let $z \in f^{-1}(\hat{E})$ be arbitrary. Take $u \in \text{PSH}(C^n)$. The formula

$$v(w) = \max\{u(z): z \in f^{-1}(w)\}, \quad w \in D_1,$$

defines a plurisubharmonic function, by Lemma 4.1. Since

$$u(z) \leq v(f(z)) \leq \sup\{v(w): w \in E \cap D_1\} = \sup\{u(s): s \in f^{-1}(E)\}$$

it follows that $z \in (\widehat{f^{-1}(E)})_{\text{PSH}(C^n)} = \widehat{f^{-1}(E)}$.

LEMMA 4.5. *Let Ω and Ω' be open sets in C^n and let f be a proper holomorphic mapping of Ω onto Ω' . Let E be a compact set in C^n , such that $\hat{E} \cap \Omega'$ is compact. If $a \in f^{-1}(E)$ and E is L -regular at the point $f(a)$, then $f^{-1}(E)$ is L -regular at a .*

Proof. From the definition of the Siciak extremal function it follows that for every compact set $K \subset C^n$ we have

$$(\Phi) \quad \Phi(z, K) = \Phi(z, \hat{K}), \quad z \in C^n.$$

Hence the set E is L -regular at the point $f(a)$ if and only if the set \hat{E} is L -regular at $f(a)$. Analogously, the L -regularity of the set $f^{-1}(E)$ at a is equivalent to the L -regularity of the set $\widehat{f^{-1}(E)}$ at a . But $\widehat{f^{-1}(E)} = (\widehat{f^{-1}(\hat{E})})$, by Lemma 4.4. Consequently, by virtue of (Φ) , we may assume that the set E is polynomially convex. By virtue of Theorem 3.2 we may also assume that $E \subset \Omega'$.

Let G be a bounded neighbourhood of $\widehat{f^{-1}(E)}$. Define

$$\Omega'_1 = \Omega' \setminus f(\Omega \setminus G),$$

$$\Omega_1 = f^{-1}(\Omega'_1).$$

The set Ω'_1 is open because f is a closed mapping. Let us observe that $E \subset \Omega'_1$ and $f: \Omega_1 \rightarrow \Omega'_1$ is a proper mapping. From Lemma 4.2 it follows that the function $h(z, f^{-1}(E), \Omega_1)$ is continuous at the point a . Since $\Omega_1 \subset G$, we have

$$0 \leq h(z, f^{-1}(E), G) \leq h(z, f^{-1}(E), \Omega_1), \quad \text{for } z \in \Omega_1.$$

From this inequality it follows readily that the function $h(z, f^{-1}(E), G)$ is continuous at the point a . Hence the set $f^{-1}(E)$ is L -regular at a , by Theorem 3.2.

The following example shows that in Lemma 4.5 the assumption of compactness of the set $\hat{E} \cap \Omega'$ cannot be omitted.

Example 4.6 (see [6], Remark 3.7). Take $\Omega = \Omega' = C \setminus \{0\}$, $E = \{z: |z| = 1\} \cup \{1/2\} \subset C$ and $f(z) = z^{-1}$. The set $f^{-1}(E)$ is not L -regular at $2 \in f^{-1}(E)$.

It should be noted that in Lemma 4.5 the assumption that the mapping f is proper cannot be replaced by the assumption that the set $f^{-1}(E)$ is compact.

Example 4.7. Let $\Omega = \Omega' = \mathbb{C}^2$. The mapping f of Ω onto Ω' defined by the formula

$$f(z_1, z_2) = (z_1, z_1 z_2^2 - z_2), \quad (z_1, z_2) \in \mathbb{C}^2$$

is not proper. Define

$$E = \{(w_1, w_2) \in \mathbb{C}^2: |w_1| = |w_2| = 1\} \cup \{(1/3, 0)\}.$$

The set $f^{-1}(E)$ has the form $F \cup \{(1/3, 3)\}$, where $F \subset \bar{B}((0, 0), 2)$. Therefore the set $f^{-1}(E)$ is not L -regular at $(1/3, 3) \in f^{-1}(E)$.

We pass now to the proof of Theorem 1.1. Assume that the hypotheses of the theorem are fulfilled.

Proof of Theorem 1.1. In view of Lemma 4.3 and the fact that a proper holomorphic mapping between two open sets in \mathbb{C}^n is open, we can choose open sets $\Omega, \Omega' \subset \mathbb{C}^n$ such that $f(\Omega) = \Omega'$, $f: \Omega \rightarrow \Omega'$ is proper and $f^{-1}(\hat{E}) \subset \Omega$. By Lemma 4.5 this implies that the set $f^{-1}(E)$ is L -regular at a .

Observe that in Theorem 1.1 the assumption that the set $f^{-1}(\hat{E})$ is compact is essential and cannot be replaced by the assumption that the set $f^{-1}(E)$ is compact.

Example 4.8. Take Ω, Ω', f and E as in Example 4.7. Then

$$\hat{E} = \{(w_1, w_2) \in \mathbb{C}^2: \max\{|w_1|, |w_2|\} \leq 1\}.$$

The set $f^{-1}(\hat{E})$ is not compact. The set $f^{-1}(E)$ is compact but not L -regular at the point $(1/3, 3)$.

Theorem 1.1 yields

COROLLARY 4.9. If $P \subset \mathbb{C}^n$ is a closed analytic polyhedron of order n , then P is L -regular.

Now suppose that the hypotheses of Theorem 1.2 are satisfied.

Proof of Theorem 1.2. Let r be a positive number. The set $f^{-1}(f(a))$ is a compact analytic subvariety in \mathbb{C}^n . Hence the set $f^{-1}(f(a))$ is finite. We may write

$$f^{-1}(f(a)) = \{a_1, \dots, a_m\}$$

where $a_j \in \mathbb{C}^n$ for $j = 1, 2, \dots, m$, $a_1 = a$ and $a_i \neq a_j$ if $i \neq j$. By the same procedure as in the proof of Theorem 1.1, we can choose open sets $\Omega, \Omega' \subset \mathbb{C}^n$ so that $f^{-1}(E) \subset \Omega$, $f(\Omega) = \Omega'$ and $f: \Omega \rightarrow \Omega'$ is proper.

Take a real number s such that

$$0 < s < r$$

$$\bar{B}(a_j, s) \subset \Omega, \quad \text{for } j = 1, 2, \dots, m$$

$$\bar{B}(a_i, s) \cap \bar{B}(a_j, s) = \emptyset \quad \text{if } i \neq j.$$

The mapping $f|_{\Omega}$ is closed, therefore there is a positive number t such that

$$\bar{B}(f(a), t) \subset \Omega' \setminus f\left(\Omega \setminus \bigcup_{j=1}^m B(a_j, s)\right).$$

Thus we have

$$f^{-1}(\bar{B}(f(a), t)) \subset \bigcup_{j=1}^m \bar{B}(a_j, s).$$

Hence the set $F = f^{-1}(E \cap \bar{B}(f(a), t)) \subset \bigcup_{j=1}^m F_j$, where $F_j = f^{-1}(E \cap \bar{B}(a_j, s))$ for $j = 1, 2, \dots, m$. The definition of the Siciak extremal function together with Theorem 1.1 shows that

$$1 \leq \lim_{z \rightarrow a} \Phi(z, \bigcup_{j=1}^m F_j) \leq \lim_{z \rightarrow a} \Phi(z, F) = 1.$$

By virtue of Theorem 3.2, the function $\Phi(z, \bigcup_{j=1}^m F_j)$ is continuous at a if and only if the function $\Phi(z, F_1)$ is continuous at a . Therefore we have

$$1 \leq \lim_{z \rightarrow a} \Phi(z, f^{-1}(E) \cap \bar{B}(a, r)) \leq \lim_{z \rightarrow a} \Phi(z, F_1) = 1.$$

Summarizing, we have proved that for every $r > 0$ the function $\Phi(z, f^{-1}(E) \cap \bar{B}(a, r))$ is continuous at a . This completes the proof.

Assume that the hypotheses in Theorem 1.3 are fulfilled.

Proof of Theorem 1.3. We may assume that the domain of f is bounded. In a similar way as in the proof of Lemma 4.4 we can show that $\widehat{f(E)} = \widehat{f(\hat{E})}$. Hence, by virtue of (Φ) , it is no restriction to assume that E is polynomially convex.

Let G be a bounded neighbourhood of $\widehat{f(E)}$. From Theorem 3.2 it follows that the function $h(z, E, f^{-1}(G))$ is continuous at a . It may easily be verified that

$$0 \leq h(f(z), f(E), G) \leq h(z, E, f^{-1}(G)), \quad \text{for } z \in f^{-1}(G).$$

Therefore the function $h(w, f(E), G)$ is continuous at $f(a)$, because the mapping f is open in a neighbourhood of a .

Hence the set $f(E)$ is L -regular at $f(a)$, by Theorem 3.2.

Theorem 1.3 yields

COROLLARY 4.10. *Let E be a compact set in \mathbb{C}^n . Let f be a holomorphic mapping defined in a neighbourhood of E , with values in \mathbb{C}^m ($m \leq n$). If E is locally L -regular at a point $a \in E$ and f is open in a neighbourhood of a then $f(E)$ is locally L -regular at the point $f(a)$.*

The last corollary is a special case of a Pleśniak result (see [6], Theorem 3.12).

5. Extremal functions and polynomial mappings. In this section we shall prove Theorem 1.4.

LEMMA 5.1. *Let $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping of degree β and let α be a positive number such that $\liminf_{|z| \rightarrow +\infty} (|f(z)|/|z|^\alpha) > 0$. If $u \in L$ and $E \subset \mathbb{C}^n$ then*

- (1) $(1/\beta)(u \circ f) \in L$,
- (2) $f(\mathbb{C}^n) = \mathbb{C}^n$ and $v_u(w) = \alpha \max\{u(z): z \in f^{-1}(w)\} \in L$,
- (3) $\alpha V(z, f^{-1}(E)) \leq V(f(z), E) \leq \beta V(z, f^{-1}(E)), \quad z \in \mathbb{C}^n$.

Proof. Property (1) follows directly from the definitions. Now we show (2). Since $\liminf_{|z| \rightarrow +\infty} (|f(z)|/|z|^\alpha) > 0$, we have

$$|z|^\alpha \leq M |f(z)|$$

for each $z \notin B(0, r)$, with appropriate constants $M > 0$ and $r > 0$. Therefore the mapping f is proper. Consequently f is a surjection and the set $f^{-1}(f(\bar{B}(0, \max\{1, r\})))$ is compact.

We define

$$\begin{aligned} \gamma_1 &= \sup \{u(z) : z \in f^{-1}(f(\bar{B}(0, \max\{1, r\})))\}, \\ \gamma_2 &= \sup \{u(z) - \log^+ |z| : z \in \mathbb{C}^n\}, \\ \gamma &= \max \{\gamma_1, \gamma_2\}. \end{aligned}$$

The constant γ_1 is finite because the function u is uppersemicontinuous. Since $u \in L$, the constant γ_2 is also finite.

Let $w \notin f(\bar{B}(0, \max\{1, r\}))$. We have the following estimates:

$$\begin{aligned} v_u(w) &\leq \alpha \gamma + \max \{\alpha \log^+ |z| : z \in f^{-1}(w)\} \\ &= \alpha \gamma + \max \{\log |z|^\alpha : z \in f^{-1}(w)\} \leq \alpha \gamma + \log M |w| \\ &\leq \alpha \gamma + \log M + \log^+ |w|. \end{aligned}$$

If $w \in f(\bar{B}(0, \max\{1, r\}))$, then $v_u(w) \leq \alpha \gamma$. Thus

$$\sup \{v_u(w) - \log^+ |w| : w \in \mathbb{C}^n\} < +\infty.$$

This fact together with Lemma 4.1 shows that $v_u \in L$.

Take $u \in L$, such that $u \leq 0$ on $f^{-1}(E)$. We deduce from (2) that $v_u \in L$ and $v_u \leq 0$ on E . Hence $v_u(w) \leq V(w, E)$ for each $w \in \mathbb{C}^n$. We have

$$u(z) \leq \max \{u(s) : s \in f^{-1}(f(z))\} = \frac{1}{\alpha} v_u(f(z)) \leq \frac{1}{\alpha} V(f(z), E), \quad z \in \mathbb{C}^n.$$

Therefore $\alpha V(z, f^{-1}(E)) \leq V(f(z), E)$ for all $z \in \mathbb{C}^n$. The second inequality in (3) follows readily from (1).

Remark 5.2. It should be noted that if $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a proper polynomial mapping then there exists a positive number α such that $\liminf_{|x| \rightarrow +\infty} (|f(x)|/|x|^\alpha) > 0$. This fact may be derived from a theorem contained in [5].

LEMMA 5.3. *If $f = (f_1, \dots, f_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial mapping and there exists $\alpha > 0$ such that $\liminf_{|z| \rightarrow +\infty} (|f(z)|/|z|^\alpha) > 0$ then $\alpha \leq \min \{\deg f_j : 1 \leq j \leq n\}$.*

Proof. Fix $j \in \{1, 2, \dots, n\}$. We can find constants $M_1 > 0$, $M_2 > 0$ and $r > 0$ such that

$$|z|^\alpha \leq M_1 |f(z)| \quad \text{for } z \in \mathbb{C}^n \setminus B(0, r)$$

and

$$|f_j(z)| \leq M_2 |z|^{\deg f_j} \quad \text{for } z \in \mathbb{C}^n.$$

