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Local triviality of a bundle of geometric objects

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Summary. Let M, E be differentiable manifolds and $\pi\colon E \to M$ be a differentiable surjection. We suppose that for some pseudogroup Γ of local transformations of M there is defined an operation of lifting of transformations of Γ to transformations of E. In this note we prove that if Γ is transitive at each point of M and Γ is transitive then $\pi\colon E \to M$ is a locally trivial bundle.

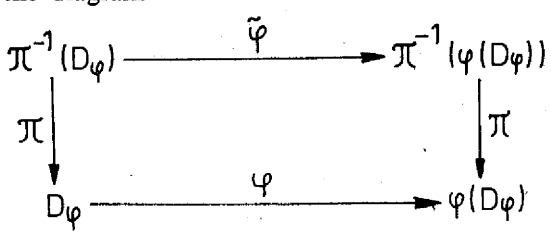
Differentiability always means that of class C^∞.

Let M be a manifold and Γ be a pseudogroup of local diffeomorphisms of M. If φ is an element of Γ we denote by D_φ the domain of φ. D_φ and $\varphi(D_\varphi)$ are open subsets of M. Let E be a manifold and $\pi\colon E \to M$ be a surjective mapping of class C^∞. A *Γ-geometry* on $\pi\colon E \to M$ is a mapping such that to each transformation φ of Γ there corresponds a transformation

$$\tilde{\varphi}\colon \pi^{-1}(D_\varphi) \to \pi^{-1}(\varphi(D_\varphi))$$

in such a way that the following conditions are satisfied:

(A) for every φ the diagram



commutes,

(B) for two elements φ and ψ of Γ, $\widetilde{\varphi \circ \psi} = \tilde{\varphi} \circ \tilde{\psi}$,

(C) $\widetilde{id}_M = id_E$,

(D) if φ is an element of Γ and U is an open subset of D_φ then

$$\widetilde{\varphi|U} = \tilde{\varphi}|\pi^{-1}(U),$$

(E) if $\Phi\colon K \to M$ is a differentiable mapping (where K is an open subset of $R \times M$) such that for all $t \in R$

$$\Phi_t\colon K_t \to M, \quad \Phi_t(x) = \Phi(t, x)$$

belongs to Γ, where $K_t = \{x \in M\colon (t, x) \in K\}$, then the mapping

$$(t, e) \to \tilde{\Phi}_t(e)$$

is of class C^∞ on its domain (which is an open subset of $R \times E$).

This definition of Γ-geometry has been given by A. Zajtz.

If $\Gamma = \Gamma_0$ is the pseudogroup of all local diffeomorphisms of M then $\pi\colon E \to M$ with some Γ_0-geometry is called a *bundle of geometric objects* (see [3]) or a *natural bundle*.

A pseudogroup Γ is called *transitive* if for two any points x and y of M there is a transformation φ of Γ such that $\varphi(x) = y$.

We denote by $L(\Gamma)$ the set of all local vector fields on M (that is, vector fields defined on open subsets of M) such that if φ_t is a local 1-parameter group of transformations of X then φ_t is an element of Γ for all t.

A pseudogroup Γ is called *transitive at a point* x of M if the tangent space T_xM is spanned by the set

$$\{X_x\colon X \in L(\Gamma), X \text{ is defined at } x\}.$$

We shall need the following lemma (see [1] and [2], p. 183).

LEMMA. *Let Γ be transitive at a point x_0 and $X_1, \ldots, X_n$ be elements of $L(\Gamma)$ such that $X_1(x_0), \ldots, X_n(x_0)$ is a basis of $T_{x_0}M$. If we denote by $\varphi_t^{(i)}$ a local 1-parameter group of transformations of X_i in some neighbourhood of x_0 then there is $\varepsilon > 0$ such that*

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \ni (t_1, \ldots, t_n) \to (\varphi_{t_1}^{(1)} \circ \ldots \circ \varphi_{t_n}^{(n)})(x_0) \in M$$

is a diffeomorphism of $(-\varepsilon, +\varepsilon)^n$ onto some open neighbourhood of x_0.

From this lemma it follows (see [1] and [2]).

COROLLARY. *If Γ is a transitive pseudogroup at each point of a connected manifold M then Γ is transitive.*

We shall now prove the following theorem.

THEOREM. *If Γ is transitive at each point of M and transitive on M and some Γ-geometry is defined on $\pi\colon E \to M$ then $\pi\colon E \to M$ is a locally trivial fibre bundle.*

Proof. Since $\pi\colon E \to M$ is surjective, by Sard's theorem there is a non-critic value x_0 of π. We fix such a point x_0 of M. Now $\pi^{-1}(x_0)$ is a submanifold of E and $\dim \pi^{-1}(x_0) = \dim E - \dim M$. Let $X_1, \ldots, X_n$ be elements of $L(\Gamma)$ such that $X_1(x_0), \ldots, X_n(x_0)$ is a basis of $T_{x_0}M$, and let $\varphi_t^{(i)}$ be a local 1-parameter group of transformations of X_i in a neighbourhood of x_0. By the Lemma, there are $\varepsilon > 0$ and an open neighbourhood U of x_0 such that

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \ni (t_1, \ldots, t_n) \to (\varphi_{t_1}^{(1)} \circ \ldots \circ \varphi)_{t_n}^{(n)}(x_0) \in U$$

is a diffeomorphism. We can suppose that $\varphi_t^{(i)}$ is defined on U for all $t \in (-\varepsilon, +\varepsilon)$ and $i = 1, \ldots, n$. Let

$$U \ni x \to (t_1(x), \ldots, t_n(x)) \in (-\varepsilon, +\varepsilon)^n$$