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Geometries defined by pseudogroups of transformations

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Summary. Let Γ be a pseudogroup of transformations of M and let $\pi\colon E \to M$ be a fibred manifold. If for each transformation $\varphi\colon U \to V$ from a pseudogroup Γ, a fibred mapping $\tilde{\varphi}\colon E|U \to E|V$ is defined in such a way that the following three conditions are satisfied: (1) $\tilde{\varphi}$ covers φ, (2) $\widetilde{\varphi \circ \psi} = \tilde{\varphi} \circ \tilde{\psi}$, $\widetilde{\mathrm{id}}_M = \mathrm{id}_E$, and (3) if $\varphi_1|W = \varphi_2|W, \varphi_1, \varphi_2 \in \Gamma$ then $\tilde{\varphi}_1|\pi^{-1}(W) = \tilde{\varphi}_2|\pi^{-1}(W)$, then we say that the correspondence $\varphi \to \tilde{\varphi}$ defines a Γ-geometry on $\pi\colon E \to M$. This definition of Γ-geometry was introduced by A. Zajtz. In this note we state some basic properties of Γ-geometries.

1. Let M be a differentiable manifold and let Γ be a pseudogroup of local diffeomorphisms of M. If φ is an element of Γ then we denote by D_φ the domain of φ. D_φ is an open subset of M and $\varphi\colon D_\varphi \to \varphi(D_\varphi)$ is a diffeomorphism of D_φ onto $\varphi(D_\varphi)$. $\varphi(D_\varphi)$ is also open in M. If φ and ψ are two elements of Γ then

$$D_{\varphi\circ\psi} = \{x \in D_\psi\colon \psi(x) \in D_\varphi\} = \psi^{-1}(D_\varphi)\,.$$

Let $\pi\colon E \to M$ be a surjective mapping of the class C^∞. (The differentiability always means the class C^∞.) If x is a point of M then $\pi^{-1}(x)$ is called a *fibre* of π. A *Γ-geometry* on E is a mapping which to each transformation φ of Γ there corresponds a transformation

$$\tilde{\varphi}\colon \pi^{-1}(D_\varphi) \to \pi^{-1}(\varphi(D_\varphi))$$

in such a way that the following conditions are satisfied:

(A) for each φ, $\tilde{\varphi}$ covers φ, i.e., the following diagram

$$\begin{tikzcd}
\pi^{-1}(D_\varphi) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \pi^{-1}(\varphi(D_\varphi)) \arrow[d, "\pi"] \\
D_\varphi \arrow[r, "\varphi"] & \varphi(D_\varphi)
\end{tikzcd}$$

commutes,

(B) for two elements φ and ψ of Γ, $\widetilde{\varphi \circ \psi} = \tilde{\varphi} \circ \tilde{\psi}$,

(C) $\widetilde{id}_M = id_E$,

(D) if φ and ψ are two elements of Γ such that $\varphi|U = \psi|U$ for some open subset U of $D_\varphi \cap D_\varphi$, then

$$\tilde{\varphi}|\pi^{-1}(U) = \tilde{\psi}|\pi^{-1}(U)\,.$$

If $\pi: E \to M$ is some special kind of fibred manifold (for example, E is a vector bundle, a principal fibre bundle, and so on) we assume that $\tilde{\varphi}$ is a morphism of structures defined on the fibres of $\pi: E \to M$. For instance, if $\pi: E \to M$ is a vector bundle then we assume that $\tilde{\varphi}$ is linear on each fibre.

We fix a Γ-geometry on $\pi: E \to M$. It is obvious that for every element φ of Γ, $\tilde{\varphi}$ is a diffeomorphism and

$$\widetilde{\varphi^{-1}} = (\tilde{\varphi})^{-1} .$$

We define $\tilde{\Gamma} = \{\tilde{\varphi}: \varphi \in \Gamma\}$ and for a point x of M

$$\Gamma_x = \{\varphi \in \Gamma: \varphi(x) = x\} .$$

Γ_x is a pseudogroup of transformations of M and $\tilde{\Gamma}$ is a pseudogroup of transformations of E. If φ is an element of Γ and x is a point of D_φ then we denote by $\tilde{\varphi}_x$ the restriction $\tilde{\varphi}_x = \tilde{\varphi}|E_x$, where $E_x = \pi^{-1}(x)$ is a fibre. $\tilde{\varphi}_x$ is a diffeomorphism of E_x onto $E_{\varphi(x)}$. In particular, if φ is an element of Γ_x then $\tilde{\varphi}_x$ is a diffeomorphism of E_x onto itself. Thus

$$G_x = \{\tilde{\varphi}_x: \varphi \in \Gamma_x\}$$

is a group. It is subgroup of the group $\mathrm{Diff}(E_x)$ of all diffeomorphisms of E_x. The group G_x is called a *structural group* of the Γ-geometry at x. We have the following

PROPOSITION 1. *If x and y are two points of M for which there is an element ψ of Γ such that $\psi(x) = y$, then the groups G_x and G_y are isomorphic. The isomorphism of these groups is given by the formula*

$$G_x \ni \tilde{\varphi}_x \to (\widetilde{\psi \circ \varphi \circ \psi^{-1}})_y \in G_y .$$

The proof is trivial.

A pseudogroup Γ of transformation of M is called *transitive* on M if for each of two points x and y of M there is an element φ of Γ such that $\varphi(x) = y$. From Proposition 1 we obtain

COROLLARY 1. *If Γ is a transitive pseudogroup on M then all structural groups G_x are isomorphic.*

Let $\sigma: M \to E$ be a section of E, i.e., $\pi \circ \sigma = id_M$. σ is called *Γ-invariant* if for each element φ of Γ and for each point x of D_φ

$$\tilde{\varphi}(\sigma(x)) = \sigma(\varphi(x)) .$$

Invariant sections were considered in [3]. We have the following properties of invariant sections.

PROPOSITION 2. *If Γ is transitive and $\sigma_1, \sigma_2: M \to E$ are two Γ-invariant sections such that $\sigma_1(x_0) = \sigma_2(x_0)$ for some point x_0 of M, then $\sigma_1 = \sigma_2$.*

Proof. For a point x of M, let φ be such an element of Γ that $\varphi(x_0) = x$. Now, for $i = 1, 2$, we have

$$\sigma_i(x) = \sigma_i(\varphi(x_0)) = \tilde{\varphi}(\sigma_i(x_0)) .$$

Proposition 3. *If $\sigma\colon M \to E$ is a Γ-invariant section, then for each point x of M and for each element A of the structural group G_x*

$$A(\sigma(x)) = \sigma(x)\,,$$

that is, the group G_x stabilizes the point $\sigma(x)$.

Proof. If $A = \tilde{\varphi}|E_x$, where φ is an element of Γ_x then

$$A(\sigma(x)) = \tilde{\varphi}(\sigma(x)) = \sigma(\varphi(x)) = \sigma(x)\,.$$

Proposition 4. *If Γ is transitive and p_0 is a point of a fibre $\pi^{-1}(x_0)$ such that the structural group G_{x_0} stabilizes p_0, then there is one and only one Γ-invariant section $\sigma\colon M \to E$ such that $\sigma(x_0) = p_0$ (*).*

Proof. If x is a point of M then we choose an element φ of Γ such that $\varphi(x_0) = x$ and we define

$$\sigma(x) = \tilde{\varphi}(p_0)\,.$$

$\sigma(x)$ is well-defined, that is, $\sigma(x)$ is independent of the choice of φ, because if φ and $\bar{\varphi}$ are two transformations such that $\varphi(x_0) = \bar{\varphi}(x_0) = x$, then $\varphi^{-1} \circ \bar{\varphi}$ belongs to Γ_{x_0} and

$$A = \widetilde{(\varphi^{-1} \circ \bar{\varphi})}_{x_0} \in G_{x_0}$$

stabilizes p_0. Thus

$$p_0 = A(p_0) = (\tilde{\varphi})^{-1}(\tilde{\bar{\varphi}}(p_0))\,.$$

This implies that $\tilde{\varphi}(p_0) = \tilde{\bar{\varphi}}(p_0)$. Now, σ is a Γ-invariant section of E such that $\sigma(x_0) = p_0$.

In the case of vector bundle $\pi\colon E \to M$, from Propositions 2 and 3 we obtain

Corollary 2. *If $\pi\colon E \to M$ is a vector bundle with some Γ-geometry then we have:*

(a) *If Γ is transitive on M and σ is a Γ-invariant section which is zero at some point of M then σ vanishes identically on M.*

(b) *If a Γ-geometry on E admits a non-zero Γ-invariant section, then for each point x of M there is a non-zero vector in E_x which is an eigenvector of all $A \in G_x$ with an eigenvalue equal to* 1.

(c) *If a Γ-geometry on E is such that at some point x_0 of M the structural group G_{x_0} contains an automorphism for which* 1 *is not an eigenvalue, then this Γ-geometry does not admit a non-zero Γ-invariant section.*

Next we come back to our general considerations, that is, a Γ-geometry is defined on an arbitrary fibred manifold $\pi\colon E \to M$.

Two sections $\sigma_1, \sigma_2\colon M \to E$ are called *Γ-conjugate* if for each point x of M there is an element A_x of the structural group G_x such that

$$\sigma_2(x) = A_x(\sigma_1(x))\,.$$

From Proposition 2 we obtain

Proposition 5. *If two sections $\sigma_1, \sigma_2\colon M \to E$ such that $\sigma_1(x_0) = \sigma_2(x_0)$ for some point x_0 of M are Γ-conjugate and σ_1 is Γ-invariant, then $\sigma_1 = \sigma_2$.*

(*) In general, σ is not continuous, see the example in this note.

2. We shall introduce infinitesimal Γ-transformations. A vector field X on M is called an *infinitesimal Γ-transformation* on M if each local 1-parameter group φ_t of transformations of X is such that for all t, $|t|<\varepsilon$, φ_t belongs to Γ. Let $L(\Gamma)$ denote the set of all infinitesimal Γ-transformations on M. Using the arguments of S. E. Salvioli it is easy to show

PROPOSITION 6. *$L(\Gamma)$ is a vector space.*

In general, $L(\Gamma)$ is not a Lie algebra.

A Γ-geometry on a fibred manifold $\pi\colon E \to M$ is called *regular* if for each differentiable mapping $\varphi\colon (-\varepsilon, +\varepsilon)\times U \to M$, $\varphi(t, x) = \varphi_t(x)$, such that φ_t belongs to Γ for all t, $|t|<\varepsilon$, the mapping

$$(-\varepsilon, +\varepsilon)\times E|U \ni (t, p) \to \tilde{\varphi}_t(p) \in E$$

is of class C^∞ (as the mapping of two variables).

For a regular Γ-geometry on E we define a *lifting of infinitesimal Γ-transformations* from M to E in the natural way. If X is an infinitesimal Γ-transformation on M then X induces on some neighbourhood U a local 1-parameter group φ_t of transformations such that φ_t belongs to Γ for all t, $|t|<\varepsilon$. Now, $\tilde{\varphi}_t$ is a local 1-parameter group of transformations on $\pi^{-1}(U)$ and let $\tilde{X}$ be a vector field on $\pi^{-1}(U)$ induced by $\tilde{\varphi}_t$. If φ_t and φ'_t are two local 1-parameter groups of transformations induced by the same vector field X respectively on U and on U', and $\tilde{X}$ and $\tilde{X}'$ are vector fields induced respectively by $\tilde{\varphi}_t$ and $\tilde{\varphi}'_t$, then $\tilde{X} = \tilde{X}'$ on $\pi^{-1}(U \cap U')$. Thus for an infinitesimal Γ-transformation X on M we have obtained a global vector field $\tilde{X}$ on E. $\tilde{X}$ is an infinitesimal $\tilde{\Gamma}$-transformation. Using the arguments of S. E. Salvioli [2] we have

PROPOSITION 7. *If X, Y are infinitesimal Γ-transformations and a, b are real numbers, then*

$$\widetilde{(aX+bY)} = a\tilde{X}+b\tilde{Y}.$$

Now we shall give an example of a Γ-geometry which is not regular. Let Γ be the group of translations on R, that is

$$\Gamma = \{f_a\colon\ a \in R\},$$

where $f_a(x) = x+a$, $x \in R$. We define a Γ-geometry on the trivial vector bundle $E = R\times R$ by the formula

$$f_a\colon\ E \ni (x, y) \to \big(x+a,\ \varepsilon(a)y\big) \in E,$$

where $\varepsilon\colon R \to R-\{0\}$ is a solution of the equation

$$\varepsilon(t_1+t_2) = \varepsilon(t_1)\varepsilon(t_2).$$

If we choose a non-continuous solution ε then we obtain a Γ-geometry on E which is not regular. Let us remark that Γ-invariant sections of this geometry are defined by the formula

$$\sigma(x) = \big(x,\ \varepsilon(x)y_0\big),$$

where y_0 is a real constant. σ is not continuous.

We say that a pseudogroup Γ is *transitive at a point* x of M if the set

$$W_x = \{X_x\colon X \in L(\Gamma)\}$$

generates the tangent space T_xM. We shall now prove two propositions.

Proposition 8. *Let Γ be a transitive pseudogroup at each point of a connected manifold M. If $\sigma_1, \sigma_2\colon M \to E$ are two Γ-invariant sections of E and $\sigma_1(x_0) = \sigma_2(x_0)$ for some point x_0 of M, then $\sigma_1 = \sigma_2$.*

Proposition 9. *If Γ is a transitive pseudogroup at each point of M and a Γ-geometry on E is regular, then every Γ-invariant section of E is of class C^∞.*

The proofs of these propositions are based on the following lemma (see [4]).

Lemma. *If a pseudogroup Γ is transitive at a point x_0 of M and $X_1, \ldots, X_n$ are infinitesimal Γ-transformations defined in some neighbourhood of x_0 such that $X_1(x_0), \ldots, X_n(x_0)$ form a basis of $T_{x_0}M$, then there is $\varepsilon>0$ such that*

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \ni (t_1, \ldots, t_n) \to (\varphi^1_{t_1} \circ \ldots \circ \varphi^n_{t_n})(x_0) \in M$$

is a diffeomorphism, where φ^i_t denotes a local 1-parameter group of transformations of X_i in a neighbourhood of x_0.

We shall now prove Proposition 8. In order to do this, let

$$M_0 = \{x \in M\colon \sigma_1(x) = \sigma_2(x)\}.$$

M_0 is a non-empty and closed subset of M_0. Let x be a point of M_0 and let

$$U = \Phi((-\varepsilon, +\varepsilon)^n),$$

where Φ and ε are such as in Lemma. This lemma implies that for any point y of U there is an element ψ of Γ such that $\psi(x) = y$. Now

$$\sigma_i(y) = \sigma_i(\psi(x)) = \tilde{\psi}(\sigma_i(x))$$

for $i = 1, 2$, and hence $U \subset Y$, i.e., M_0 is also open. By the connectivity, we obtain that $M = M_0$ and the proof is finished.

Next we shall prove Proposition 9. Let σ be a Γ-invariant section of E. For a point x of M, by Lemma, there are a neighbourhood U of x and $\varepsilon>0$ such that

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \to U$$

is a diffeomorphism. We write

$$(t_1(y), \ldots, t_n(y)) = \Phi^{-1}(y).$$

The functions $t_1, \ldots, t_n$ are of class C^∞ on U and from the equality

$$y = (\varphi^n_{t_n(y)} \circ \ldots \circ \varphi^1_{t_1(y)})(x)$$

it follows that

$$\sigma(y) = (\tilde{\varphi}^n_{t_n(y)} \circ \ldots \circ \tilde{\varphi}^1_{t_1(y)})(\sigma(x)).$$

Thus σ is of class C^∞ on U.

3. We fix a regular Γ-geometry on $\pi\colon E \to M$. We shall define a "Lie derivation" of sections of E in the direction of infinitesimal Γ-transformations. We denote by $VE \to E$ the vector bundle of vertical vectors on E, that is, for a point e of E, $V_eE = \ker d_e\pi$. Let $\sigma\colon M \to E$ be a section and X be an infinitesimal Γ-transformation. If x is a point of M, then the curve

$$t \to \tilde{\varphi}_t\big(\sigma(\varphi_{-t}(x))\big),$$

where φ_t is a local 1-parameter group of transformations of X, is vertical because it lies in $\pi^{-1}(x)$. Let $(L_X\sigma)(x)$ be the velocity vector of this curve for $t = 0$. $(L_X\sigma)(x)$ belongs to $V_{\sigma(x)}E$ and the mapping

$$L_X\sigma\colon\ M \ni x \to (L_X\sigma)(x) \in VE$$

is a section of $p\colon VE \to M$, where p is the composition of $\pi\colon E \to M$ and the bundle projection $VE \to E$. $L_X\sigma$ is called the *Lie derivative of σ in the direction of X*. The definition of $L_X\sigma$ implies that the following diagram commutes, where $VE \to E$ is the bundle projection.

$$\begin{tikzcd}
& VE \arrow[d] \\
M \arrow[ur, "L_X\sigma"] \arrow[r, "\sigma"] & E
\end{tikzcd}$$

If E is a vector bundle, then $VE \to E$ is isomorphic with the pull-back of E by $\pi\colon E \to M$. Hence, in this case $L_X\sigma$ is uniquely determined by some section of $\pi\colon E \to M$ denoted by $\mathscr{L}_X\sigma$. $\mathscr{L}_X\sigma$ is also called the *Lie derivative of σ in the direction of X*, but $\mathscr{L}_X\sigma$ is defined only in the case of vector bundles. If $E = TM$ is a tangent bundle and a Γ_0-geometry on TM is defined by the formula $\tilde{\varphi} = d\varphi$, then $\mathscr{L}_X\sigma$ is the standard Lie derivative of vector fields. (Γ_0 denotes the pseudogroup of all local diffeomorphisms of M.)

We must stress that L_X (or $\mathscr{L}_X$ in the case of a vector bundle) is defined only if X is an infinitesimal Γ-transformation.

From Proposition 7 we obtain

PROPOSITION 10. *If X, Y are infinitesimal Γ-transformations and a, b are real numbers, then*

$$L_{aX+bY}\sigma = aL_X\sigma + bL_Y\sigma .$$

The linear combination $aL_X\sigma + bL_Y\sigma$ is defined in a natural manner:

$$(aL_X\sigma + bL_Y\sigma)(x) = a(L_X\sigma)(x) + b(L_Y\sigma)(x) .$$

This is possible because $(L_X\sigma)(x)$ and $(L_Y\sigma)(x)$ belong to $V_{\sigma(x)}E$.

PROPOSITION 11. *If E is a vector bundle then for all infinitesimal Γ-transformations X, Y, all real numbers a, b and every function f on M we have*

$$\mathscr{L}_{aX+bY}\sigma = a\mathscr{L}_X\sigma + b\mathscr{L}_Y\sigma$$

$$\mathscr{L}_X(\sigma_1+\sigma_2) = \mathscr{L}_X\sigma_1 + \mathscr{L}_X\sigma_2 , \qquad \mathscr{L}_X(f\sigma_1) = (Xf)\sigma_1 + f\mathscr{L}_X\sigma_1 .$$

According to the definitions of L and L we have (cf. [3])

Proposition 12. *Let $\sigma\colon M \to E$ be a section. $L_X\sigma = 0$ ($\mathscr{L}_X\sigma = 0$ in the case of a vector bundle) for every infinitesimal Γ-transformation X if and only if σ is invariant with respect to every local 1-parameter group of Γ-transformations of M. If a section σ is Γ-invariant, then $L_X\sigma = 0$ ($\mathscr{L}_X\sigma = 0$ in the case of a vector bundle) for every infinitesimal Γ-transformation X.*

4. At the end of this note we give two examples.

(I) Let M be a Riemannian space with a metric tensor g and let Γ denote the pseudogroup of local isometries of M. We define a Γ-geometry on the vector bundle $E = T^*M \otimes T^*M$. For a local isometry $\varphi\colon U \to V$ we set

$$\tilde{\varphi} = \big((d\varphi)^*\big)^{-1} \otimes \big((d\varphi)^*\big)^{-1}\colon E|U \to E|V,$$

where $(d\varphi)^*$ is the transposed mapping of $d\varphi$. The metric tensor g is a Γ-invariant section of E.

(II) Let M, g and Γ be the same as in the example (I). We define a Γ-geometry on the tangent bundle TM setting $\tilde{\varphi} = d\varphi$, for $\varphi \in \Gamma$. If Γ is transitive and $v\colon M \to TM$ is a Γ-invariant, then v has a constant length. The structural group G_x is a subgroup of $0(n, R)$. If $G_x = 0(n, R)$ for each x, then two vector fields $v_1, v_2\colon M \to TM$ are Γ-conjugate if and only if $\|v_1(x)\| = \|v_2(x)\|$ for all x. Hence, if the Γ-geometry on TM is such that $G_x = 0(n, R)$ and Γ is transitive, then this Γ-geometry does not admit a non-zero Γ-invariant section of TM.

References

[1] J. Gancarzewicz, *Local triviality of a bundle of geometric objects*, Zeszyty Nauk. UJ,

[2] S. E. Salvioli, *On the theory of geometric objects*, J. Diff. Geom. 7 (1972), 257–278.

[3] T. E. Steward, *Lifting group actions in fibre bundles*, Ann. of Math. 74 (1961), 192–198.

[4] H. I. Susman, *Orbits of families of vector fields and integrability of Distributions*, Tran. Amer. Math. Soc. 180 (1973), 173–188.

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