

Complete lifts of tensor fields of type $(1, k)$ to natural bundles

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§ 0. Introduction and notations

Let $\pi: E \rightarrow M$ be a natural bundle. For each local diffeomorphism φ of M there is a fibred local diffeomorphism $\tilde{\varphi}$ of E such that $\tilde{\varphi}$ covers φ and the well-known conditions are satisfied (the conditions (A)–(E) in § 1). Using local 1-parameter groups of transformations we can define a mapping $\mathcal{X}(M) \rightarrow \mathcal{X}(E)$ of the Lie algebra of vector fields on M into the Lie algebra of vector fields on E . If X is a vector field on M then the image of X , denoted by X^c , is called a *complete lift* of X from M to E . The mapping

$$X \rightarrow X^c$$

is a Lie algebra homomorphism (Proposition 1.1, see also [14]). This proposition generalizes the analogous results given in [7]–[10], [18]–[22]. Here we prove this general property by a homogeneous method using the arguments of S. E. Salvioli [14].

Let t be a tensor field of type $(1, k)$ on M . We say that t admits a *complete lift* to E if there is a tensor field t^c , called a *complete lift* of t , of type $(1, k)$ on E such that

$$t^c(X_1^c, \dots, X_k^c) = (t(X_1, \dots, X_k))^c$$

for all vector fields X_1, \dots, X_k on M . In the present paper we state some properties of complete lifts of tensor fields of type $(1, k)$ and we discuss problems of the existence of a complete lift of tensor fields of type $(1, k)$ from M to E .

This definition of complete lift generalizes the definitions given by K. Yano and S. Kobayashi [20]–[22], K. Yano and S. Ishihara [18], [19] and A. Morimoto [7]–[10] in cases of natural bundles of special kinds such as tangent bundles, tangent bundles of higher order, tangent bundles of p^* -velocities and bundles of infinitesimal near points. Our definition does not coincide with that of K. Yano and E. M. Patterson [23] in the case of cotangent bundles (see (14.3) and (15.3) in [23]). We shall prove that in the case of cotangent bundles, tensor fields do not admit complete lifts in the sense of our definition (Proposition 6.2) instead of tensor fields written in the form $f\delta$, where f is a function on M .

In § 2 we give a classification of natural bundles which follows from the results of R. S. Palais and C.-L. Terng [13]. This classification (Theorem 2.1) shows that every

natural bundle over M is isomorphic to an associated fibre bundle with the r -frame bundle $F^r M$.

In § 3 we study a complete lift of tensor fields from M to the r -frame bundle $F^r M$. These results follow from those obtained by A. Morimoto [7], [8] because $F^r M$ is an open subset of the tangent bundle of n^r -velocities, where $n = \dim M$. In the case of r -frame bundles, using an isomorphism given in Proposition 2.1, we can simplify A. Morimoto's arguments to show the basic properties of a complete lift of tensor fields (Proposition 3.3).

Let y be a point of a natural bundle E . We denote by W_y the subspace of the tangent space $T_y E$ of all vectors written in the form $X^c(y)$, where X is a vector field on M , and let

$$\begin{aligned} E_0 &= \{y \in E: \dim W_y = \dim E\} \\ &= \{y \in E: W_y = T_y E\}. \end{aligned}$$

We prove (Proposition 1.3) that E_0 is an open subset of E and $\pi: E_0 \rightarrow M$ is also a natural bundle if E_0 is not empty.

In §§ 4 and 5 we always suppose that E_0 is dense in E . This assumption is necessary to ensure the uniqueness of complete lifts of tensor fields of type $(1, k)$. This assumption is verified in the cases of special natural bundles considered in [7]–[10], [18]–[22].

In § 4 we prove some (in general, algebraic) properties of complete lifts of tensor fields. In § 5 we give necessary and sufficient conditions under which a tensor field t admits a complete lift from M to E (Theorem 5.2). In § 6 we give some examples of the applications of our results.

Let M be a manifold. If x is a point of M then we denote by $T_x M$ the tangent space to M at x and TM denotes the tangent bundle. If $f: M \rightarrow N$ is a differentiable mapping then we denote by $df: TM \rightarrow TN$ the induced mapping of the tangent bundles. If x is a point of M , $d_x f$ denotes the restriction of df to $T_x M$. $d_x f$ is a linear mapping of $T_x M$ into $T_{f(x)} N$. We denote by $C^\infty(M)$ the ring of all differentiable functions on M and by $\mathcal{X}(M)$ the $C^\infty(M)$ -module of all differentiable vector fields on M .

In the present paper, manifolds, vector fields and so on always mean differentiable manifolds, differentiable vector fields and so on. Differentiability always means the differentiability of class C^∞ .

In this paper we shall use the Einstein summation convention.

§ 1. Natural bundles and a complete lift of tensor fields

Let M be a manifold. If U and V are open subsets of M then a diffeomorphism $\varphi: U \rightarrow V$ of U onto V is called a *local diffeomorphism* of M . The set Γ_0 of all local diffeomorphisms of M is a pseudogroup. If φ is a local diffeomorphism of M then we denote by D_φ the domain of φ . If φ and ψ are local diffeomorphisms of M then

$$D_{\psi \circ \varphi} = \{x \in D_\varphi: \varphi(x) \in D_\psi\} = \varphi^{-1}(D_\psi).$$

Let $\pi: E \rightarrow M$ be a locally trivial fibre bundle. It is called a *natural bundle* if for every local diffeomorphism φ of M there is a local diffeomorphism $\tilde{\varphi}$ of E such that the following conditions are satisfied:

(A) for every φ , the diagram

$$\begin{array}{ccc} \pi^{-1}(D\varphi) & \xrightarrow{\tilde{\varphi}} & \pi^{-1}(\varphi(D\varphi)) \\ \pi \downarrow & & \downarrow \pi \\ D\varphi & \xrightarrow{\varphi} & \varphi(D\varphi) \end{array}$$

commutes,

(B) if φ and ψ are local diffeomorphisms of M then $\overline{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi}$,

(C) $\overline{id_M} = id_E$,

(D) if φ is a local diffeomorphism of M and U is an open subset of D then

$$\tilde{\varphi}|_{\pi^{-1}(U)} = \overline{(\varphi|_U)},$$

(E) if V is an open subset of $K \times M$, where K is an arbitrary manifold, and $\Phi: V \rightarrow M$ is a differentiable mapping such that for all $k \in K$

$$\Phi_k: V_k \rightarrow M, \quad \Phi_k(x) = \Phi(k, x),$$

where $V_k = \{x \in M: (k, x) \in V\}$, is a local diffeomorphism of M , then

$$(k, y) \rightarrow \tilde{\Phi}_k(y)$$

is a differentiable mapping on its domain (this domain is an open subset of $K \times E$).

$\tilde{\varphi}$ is called a *prolongation* of φ from M to a natural bundle E .

For our considerations we fix a natural bundle $\pi: E \rightarrow M$.

Let X be a vector field on M . In some neighbourhood U of any point of M , X induces a local 1-parameter group of transformations φ_t . The conditions (B), (C) and (E) imply that $\tilde{\varphi}_t$ is a local 1-parameter group of transformations on E , hence $\tilde{\varphi}_t$ induces a vector field X^c on $\pi^{-1}(U)$. If φ_t and φ'_t are two local 1-parameter groups of transformations induced by the same vector field X on U and U' respectively then, by the condition (D), the vector fields induced by $\tilde{\varphi}_t$ and $\tilde{\varphi}'_t$ on $\pi^{-1}(U)$ and $\pi^{-1}(U')$ respectively coincide on $\pi^{-1}(U \cap U')$. Thus, X induces a global vector field X^c on M .

Definition 1.1. X^c is called the *complete lift* of a vector field X from M to E .

This definition coincides with the definition of a complete lift of vector fields to tangent bundles given by K. Yano and S. Kobayashi [20]–[22] (see also [19]), with the definition of a complete lift of vector fields to cotangent bundles given to K. Yano and E. M. Patterson [23], with the definitions of lifts $X^{(2)}$ and $X^{(r)}$ to tangent bundles of order 2 and r given by K. Yano, S. Ishihara [18] and A. Morimoto [7], [9] respectively, with the definition of a lift $X^{(0, \dots, 0)}$ to tangent bundles of p^r -velocities given by A. Morimoto [7], [8] (see also,

K.-P. Mok [5]) as well as with the definition of a lift X^A to bundles of infinitesimal near points given by A. Morimoto [7], [10].

Using the methods of S. E. Salvioli (see §§ 2 and 3 in [14]; cf. also [4]) we have immediately

PROPOSITION 1.1. *The mapping*

$$\mathcal{X}(M) \ni X \rightarrow X^c \in \mathcal{X}(E)$$

is a Lie algebra homomorphism.

This proposition generalizes similar propositions proved in [5], [7]-[10], [18]-[25] in cases of natural bundles of special kinds by methods of local coordinate calculation. S. E. Salvioli has shown this proposition in a homogeneous and very elegant way [14].

For a point y of E we denote

$$(1.1) \quad V_y E = \ker d_y \pi.$$

$VE = \bigcup_y V_y E$ is the distribution of vertical vectors on E . We denote also

$$(1.2) \quad W_y = \{X^c(y) : X \in \mathcal{X}(M)\}.$$

From Proposition 1.1 it follows that W_y is a vector subspace of the tangent space $T_y E$. In general, the distribution $W = \bigcup_y W_y$ has not a constant dimension. (For instance, if $E = TM$ is a tangent bundle then $\dim W_y = n^2$ if $y \neq 0$ and $\dim W_y = n$ if $y = 0$.) It is a regular distribution in the sense used by H. I. Susman [17], i.e., each point of E has a neighbourhood V and there are vector fields X_1, \dots, X_N on V such that W_y is spanned by $X_1(y), \dots, X_N(y)$ for all $y \in V$. From Proposition 1.1 and [17] it follows that the distribution W is integrable in the sense that for each point y of E there is a submanifold K of E containing y and such that $T_z K = W_z$ for all $z \in K$. We now show

PROPOSITION 1.2. *For each point y of E we have*

$$T_y E = V_y E + W_y,$$

$$V_y E \cap W_y = \{X^c(y) : X \in \mathcal{X}(M), X(\pi(y)) = 0\}$$

Proof. If φ_t is a local 1-parameter group of transformations of a vector field X on M then $\tilde{\varphi}_t$ is a local 1-parameter group of transformations of X^c , and by the condition (A)

$$\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi.$$

This implies that

$$(1.3) \quad d_y \pi(X^c(y)) = X(\pi(y)).$$

Thus, it follows that the second formula of our proposition is true. In order to show the first formula, let V be any vector of $T_y E$. We choose a vector field X on M such that $d_y \pi(V) = X(\pi(y))$ and now, by (1.3), we have

$$d_y \pi(V - X^c(y)) = 0.$$

Thus

$$V = (V - X^C(y)) + X^C(y)$$

belongs to $V_y E + W_y$. Our proposition is verified.

Let

$$(1.4) \quad k_0 = \max \{ \dim W_y : y \in E \}$$

$$(1.5) \quad E_0 = \{ y \in E : \dim W_y = k_0 \}.$$

From Proposition 1.2 it follows that

$$(1.6) \quad \dim M \leq k_0 \leq \dim E.$$

The case of $k_0 = \dim E$ will be the most interesting. We shall prove the following proposition.

PROPOSITION 1.3. *E_0 is an open subset of E and $\pi(E_0) = M$. Furthermore, $\pi: E_0 \rightarrow M$ is a natural bundle. For a local diffeomorphism φ of M , a prolongation of φ to E_0 is the restriction to E_0 of the prolongation $\tilde{\varphi}$ of φ to E .*

Proof. Let y_0 be a point of E_0 . There are vector fields X_1, \dots, X_{k_0} on M such that the vectors

$$X_1^C(y_0), \dots, X_{k_0}^C(y_0)$$

form a basis of W_{y_0} . Hence, there is an open neighbourhood \tilde{U} of y_0 such that

$$X_1^C(y), \dots, X_{k_0}^C(y)$$

are linearly independent if y is a point of \tilde{U} , that is, $\dim W_y \geq k_0$. By (1.4) this implies that \tilde{U} is a subset of E_0 , or E_0 is open in E .

In order to prove the remaining statements of our proposition we study

$$\tilde{\varphi}_* X^C = d\tilde{\varphi} \circ X^C \circ \tilde{\varphi}^{-1},$$

where φ is a local diffeomorphism of M and X is a vector field on M . Let φ_t be a local 1-parameter group of transformations of a vector field X . By Proposition 1.7 in [3] p. 14, $\varphi \circ \varphi_t \circ \varphi^{-1}$ is a local 1-parameter group of transformations of $\varphi_* X = d\varphi \circ X \circ \varphi^{-1}$

and by Definition 1.1, $\tilde{\varphi}_t$ and $\varphi \circ \varphi_t \circ \varphi^{-1}$ are local 1-parameter groups of transformations of X^C and $(\varphi_* X)^C$ respectively. By the conditions (B) and (C), we have

$$\overline{\varphi \circ \varphi_t \circ \varphi^{-1}} = \tilde{\varphi} \circ \tilde{\varphi}_t \circ \tilde{\varphi}^{-1},$$

hence we obtain

$$(1.7) \quad \tilde{\varphi}_* X^C = (\tilde{\varphi}_* X)^C.$$

Now we show the equality $\pi(E_0) = M$. To prove this we fix a point y_0 of E_0 (by (1.5), E_0 is not empty) and let x be any point of M . There is a local diffeomorphism φ of M such that $\pi(y_0)$ belongs to the domain of φ and $\varphi(\pi(y_0)) = x$. By (1.7) we have

$$(1.8) \quad d_y \tilde{\varphi}(W_y) = W_{\tilde{\varphi}(y)},$$

hence $\tilde{\varphi}(y_0)$ belongs to E_0 . Since

$$\pi(\tilde{\varphi}(y_0)) = \varphi(\pi(y_0)) = x,$$

thus $\pi(E_0) = M$.

If φ is any local diffeomorphism of M then, by (1.8), we obtain $\tilde{\varphi}(E_0) \subset E_0$. This permits a prolongation of φ to E_0 to be defined as the restriction of $\tilde{\varphi}$ to E_0 . By [1], this implies that E_0 is a locally trivial fibre bundle, i.e., E_0 is a natural bundle over M . The proof of Proposition 1.3 is now complete.

§ 2. r -frame bundles and a classification of natural bundles

Let M be a manifold. We denote by $F^r M$ the set of all r -jets at 0 of diffeomorphisms of open neighbourhoods of 0 in R^n , $n = \dim M$, onto open subsets of M . Let $\pi: F^r M \rightarrow M$ be the target projection

$$(2.1) \quad \pi(j_0^r \gamma) = \gamma(0).$$

$\pi: F^r M \rightarrow M$ is a principal fibre bundle with the structural group L_n^r of all r -jets with the source and with the target at 0 of local diffeomorphisms of R^n . The group L_n^r acts on $F^r M$ on the right in the natural way

$$(2.2) \quad j_0^r \gamma \cdot j_0^r \xi = j_0^r(\gamma \circ \xi),$$

where $j_0^r \gamma$ and $j_0^r \xi$ belong to $F^r M$ and L_n^r respectively. If φ is a local diffeomorphism of M then we define its prolongation $\varphi^{(r)}$ to $F^r M$

$$(2.3) \quad \varphi^{(r)}: F^r M | D_\varphi \in j_0^r \gamma \rightarrow j_0^r(\varphi \circ \gamma) \in F^r M | \varphi(D_\varphi).$$

It is easy to see that the mapping $\varphi \rightarrow \varphi^{(r)}$ satisfies the conditions (A)–(E) of the definition of natural bundle. $F^r M$ is called the r -frame bundle.

Let N_r denote the set of all n -tuples $v = (v_1, \dots, v_n)$ of non-negative integers such that

$$|v| = v_1 + \dots + v_n \leq r.$$

Every chart (U, x^1, \dots, x^n) on M induces a chart

$$(\pi^{-1}(U), x^i: i = 1, \dots, n; v \in N_r)$$

on $F^r M$, called the *induced chart*, where

$$(2.4) \quad x^i(j_0^r \gamma) = \frac{1}{v!} D_v(x^i \circ \gamma)(0).$$

In future, we shall often use induced charts.

Let us observe that $F^r M$ is an open and dense subset of the tangent bundle of n^r -velocities, where $n = \dim M$ [8].

We denote by X^{Cr} the complete lift of a vector field X from M to the r -frame bundle F^rM . From Proposition 1.1 it follows

$$(2.5) \quad (aX + bY)^{Cr} = aX^{Cr} + bY^{Cr}$$

$$(2.6) \quad [X, Y]^{Cr} = [X^{Cr}, Y^{Cr}]$$

for all vector fields X, Y and all real numbers a, b . We have the following isomorphism (see [2], [6] or [12]).

PROPOSITION 2.1. *For each point p of F^rM the mapping*

$$J_{\pi(p)}^r(TM) \in J_{j(p)}^r X \rightarrow X^{Cr}(p) \in T_p(F^rM)$$

is a linear isomorphism, where $J_x^r(TM)$ denotes the space of all r -jets at x of vector fields on M .

Proof. Using an induced chart and local 1-parameter groups of transformations of X and X^{Cr} we can verify by a straightforward calculation that the mapping considered in our proposition is a well-defined monomorphism. It is an isomorphism because

$$\dim J_{\pi(p)}^r(TM) = \dim T_p(F^rM) = n \binom{n+r}{r}.$$

This remark finishes our proof.

From (2.2) and (2.3) it follows immediately that

$$(2.7) \quad \varphi \circ R_\xi = R_\xi \circ \varphi,$$

where R_ξ denotes the right translation on F^rM , hence according to Corollary 1.8 p. 14 in [3], we obtain

PROPOSITION 2.2. *If X is a vector field on M and ξ is an element of L_n^r then*

$$(R_\xi)_* X^{Cr} = X^{Cr},$$

or equivalently, for each point p of F^rM

$$(d_p R_\xi)(X^{Cr}(p)) = X^{Cr}(p \circ \xi).$$

Let F be a manifold on which the group L_n^r acts on the left and let $E = E(M, F, L_n^r, F^rM)$ denote the associated fibre bundle with the r -frame bundle F^rM , with the standard fibre F (see [3] p. 54-55). We recall that E is a manifold of all orbits of L_n^r on $F^rM \times F$, where L_n^r acts on $F^rM \times F$ as follows

$$(p, f) \circ \xi = (p \circ \xi, \xi^{-1} \circ f).$$

Let $\Phi: F^rM \times F \rightarrow E$ be the canonical mapping, i.e., $\Phi(p, f)$ is the orbit containing (p, f) . Thus

$$(2.8) \quad \Phi(p, f) = \Phi(p', f') \Leftrightarrow \exists \xi \in L_n^r: p' = p \cdot \xi, f' = \xi^{-1} \cdot f.$$

Now, $\pi_E: E \rightarrow M$, $\pi_E(\Phi(p, f)) = \pi(p)$, is a natural bundle, where a prolongation $\tilde{\varphi}$ of a local diffeomorphism $\tilde{\varphi}$ of M is defined by the formula

$$(2.9) \quad \tilde{\varphi}(\Phi(p, f)) = \Phi(\tilde{\varphi}^{(r)}(p), f).$$

and $\tilde{\varphi}^{(r)}$ is given by (2.3). $\tilde{\varphi}$ is well-defined because, by (2.7) and (2.8), we have the following implication

$$\Phi(p, f) = \Phi(p', f') \Leftrightarrow \Phi(\tilde{\varphi}^{(r)}(p), f) = \Phi(\tilde{\varphi}^{(r)}(p'), f').$$

It is obvious that the correspondence $\varphi \rightarrow \tilde{\varphi}$ satisfies the conditions (A)–(E) of the definition of natural bundles.

As in § 1, we denote by X^C a complete lift of a vector field X from M to E . We have the following relation between X^{Cr} and X^C .

PROPOSITION 2.3. *If X is a vector field on M then for each element (p, f) of $F^rM \times F$ we have*

$$(d_p \Phi_f)(X^{Cr}(p)) = X^C(\Phi(p, f)),$$

where $\Phi_f: F^rM \rightarrow E$ is given by the formula

$$(2.10) \quad \Phi_f(p) = \Phi(p, f).$$

Proof. Let φ_t be a local 1-parameter group of transformations of X in a neighbourhood of $\pi(p)$. $\tilde{\varphi}_t^{(r)}$ and $\tilde{\varphi}_t$ are local 1-parameter groups of transformations of X^{Cr} and X^C in neighbourhoods of p and $\Phi(p, f)$ respectively. From (2.9) it follows that

$$\tilde{\varphi}_t \circ \Phi_f = \Phi_f \circ \tilde{\varphi}_t^{(r)},$$

hence we obtain our proposition.

For a point f of F we denote by H_f the isotropy group of f , i.e.,

$$(2.11) \quad H_f = \{\xi \in L_n^r: f = f \cdot \xi\}.$$

If p is a point of F^rM we denote

$$(2.12) \quad K(p, f) = \{A_p^*: A \in L(H_f)\}.$$

where A^* is a fundamental vertical vector field on F^rM induced by an element A of the Lie algebra $L(H_f)$ of the Lie group H_f . $K(p, f)$ is a linear subspace of the tangent space $T_p(F^rM)$ and for any point f , the distribution $p \rightarrow K(p, f)$ is integrable because the mapping $A \rightarrow A^*$ is a Lie algebra homomorphism (see [3]). We now show the following technical equality used in § 5 of the present paper.

PROPOSITION 2.4. *If p is a point of F^rM and f is a point of F then*

$$\ker d_p \Phi_f = K(p, f),$$

where Φ_f and $K(p, f)$ are defined by (2.10) and (2.12) respectively.

Proof. Let A be an element of $L(H_f)$ and a_t be the 1-parameter subgroup of A . R_{a_t} is the 1-parameter group of transformations of the fundamental vertical vector field A^* (see [3]), hence, using (2.8) we obtain

$$\begin{aligned} (d_p \Phi_f)(A_p^*) &= \frac{d}{dt} (\Phi_f(R_{a_t}(p)))|_{t=0} \\ &= \frac{d}{dt} (\Phi(p \cdot a_t, f))|_{t=0} \\ &= \frac{d}{dt} (\Phi(p, a_t \cdot f))|_{t=0} \\ &= 0. \end{aligned}$$

This means that $\ker d_p \Phi_f \supset K(p, f)$. We need only to show

$$(2.13) \quad \ker d_p \Phi_f \subset K(p, f).$$

In order to do this, let V be a vector of $\ker d_p \Phi_f$. Since $\pi = \pi_E \circ \Phi_f$, we have

$$d\pi(V) = (d\pi_E \circ d_p \Phi_f)(V) = 0.$$

This implies that V is a vertical vector. Thus there is an element A of $L(L'_n)$ such that $V = A_p^*$. We must show that A belongs to $L(H_f)$.

We choose a local section σ of $F^r M$ defined on some neighbourhood U of $\pi(p)$ such that $\sigma(\pi(p)) = p$. This section defines two diffeomorphisms

$$(2.14) \quad \phi^r: U \times L'_n \ni (x, \xi) \rightarrow \sigma(x) \cdot \xi \in F^r M|U$$

$$(2.15) \quad \phi: U \times F \ni (x, f) \rightarrow \Phi(\sigma(x), f) \in E|U,$$

and it is easy to verify

$$(2.16) \quad (\phi^{-1} \circ \Phi_f \circ \phi^r)(x, \xi) = (x, \xi \cdot f).$$

If we denote

$$(2.17) \quad \varrho_p: L'_n \rightarrow \pi^{-1}(\pi(p)), \varrho_p(\xi) = p \cdot \xi,$$

$$(2.18) \quad \varrho_f: L'_n \rightarrow F, \varrho_f(\xi) = \xi \cdot f,$$

then, by (2.16) we have

$$(2.19) \quad (\phi^{-1} \circ \Phi_f \circ \varrho_p)(\xi) = (\pi(p), \varrho_f(\xi)).$$

By the definition of fundamental vertical vector fields, $A_p^* = (d_e \varrho_p)(A)$, where e is the identity element of L'_n , and hence, by (2.19)

$$(d_e \varrho_f)(A) = d_p(\phi^{-1} \circ \Phi_f)(A_p^*) = 0.$$

This means that A belongs to $\ker d_e \varrho_f = L(H_f)$ and the proof of (2.13) is finished.

At the end of this section we give a classification of natural bundles which follows from the results of R. S. Palais and C.-L. Terng [13].

