ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DCCXXIII PRACE MATEMATYCZNE, ZESZYT 23 1982

On the convergence of the iterated Pilgerschritt transformation

by N. Netzer

The solution of the translation equation in a real Lie-group G can be defined as the construction of a continuous homomorphism $h\colon R \to G$ where $h(1) = f$ is a given element of G (cf. R. Liedl, [5]). In order to solve this problem in a more general case and if the restriction $h/[0, 1]$ is required to be homotopic to a given path $\varphi\colon [0, 1] \to G$ from the unit element to the element f, R. Liedl has proposed a method called the Pilgerschritt-transformation. In this paper we shall give sufficient conditions for this method to solve this problem.

Since a Lie-group G is locally isomorphic to a group of matrices and our conditions will be formulated using such a local isomorphism, we shall restrict the group G to a closed subgroup of the group $Gl(n, R)$ of the real $n\times n$-matrices. Let $f \in G$ be the given $n \times n$-matrix and let the C^1-path $\varphi\colon [0, 1] \to G$ with $\varphi(0) = E$ and $\varphi(1) = f$ represent the required homotopy class of the restriction $h/[0, 1]$ of the homomorphism $h\colon R \to G$ in question. Then it is possible to consider the solution of the matrix differential equation

$$\frac{\partial\varphi(t, \tau)}{\partial t} = \tau\varphi'(t)\varphi(t)^{-1}\varphi(t, \tau)$$

with the initial condition $\varphi(0, \tau) = E$ (where $\tau \in [0, 1]$ denotes a real parameter). The path $\varphi\colon [0, 1] \to G$ defined by $\tilde{\varphi}(\tau) = \varphi(1, \tau)$ is called the Pilgerschritt transform of φ (cf. R. Liedl [5]).

The iteration of the Pilgerschritt transformation gives rise to a sequence of paths $\varphi, \tilde{\varphi}, \tilde{\tilde{\varphi}}, \ldots, \overset{n}{\tilde{\varphi}}, \ldots$ Liedl's conjecture was that there exist weak conditions for the path φ implying the uniform convergence of this sequence to a path $\chi\colon [0, 1] \to G$, which is homotopic to φ and is the restriction of a continuous homomorphism $h\colon R \to G$.

In this paper we wish to show that there exists a positive real number L such that this conjecture is true, if $\|\varphi'(t)\| < L$ for each $t \in [0, 1]$ (e.g., $\|\cdot\|$ is the operator norm). If $\|\varphi'(t)\|$ is small enough for each $t \in [0, 1]$, by integration we see that $\|\varphi(t) - E\|$ is small. Thus $\varphi(t)$ can be written in the form

$$\varphi(t) = \exp(tD)\exp\bigl(A(t)\bigr)$$

where $\exp(D) = f$ and $A(0) = A(1) = 0$.

Because of

$$\varphi'(t) = D\exp(tD)\exp(A(t)) + \exp(tD)\frac{d}{dt}\exp(A(t))$$

we have $\|\varphi'(t)\|$ small iff $\|D\|$ and $\|A'(t)\|$ are small. Therefore it is sufficient to show the following

THEOREM: *For each $K \in (0, 1)$ there exists a $\beta > 0$ such that $\|D\| < \beta_1$ and $\max_{t\in[0,1]} \|A'(t)\| < \beta_1$ and $\beta_1 < \beta$ implies that*

$$\varphi(t) = \exp(tD)\exp(A(t)),$$

where

$$\max_{t\in[0,1]} \|A'(t)\| < K\beta_1 .$$

We first prove the following

LEMMA: *Let $\delta > 0$ be such that for $X, Y \in \mathscr{L}(G)$ (we identify the Lie-algebra of G with a subalgebra of the algebra of real $n\times n$-matrices) the Campbell–Baker–Hausdorff-series*

$$H(X, Y) = \log(\exp X \exp Y) = X + Y + \tfrac{1}{2}[X, Y] + \dots$$

converges whenever $\|X\|, \|Y\| < \delta$. We put

$$F(X, Y) = \frac{d}{dt}(H(tX, Y))|_{t=0} = X + \sum_{k\geq 1} \alpha_k \underbrace{[Y, [Y, \dots [Y, X]] \dots]}_{k \text{ brackets}} (F)$$

($\alpha_k = \dfrac{\xi_k}{k!}$ where ξ_k is the k-th Bernoulli-number) and

$$g(t, \tau) = \exp((1-\tau)tD)\left(\frac{d}{dt}\exp(A(t))\right)\exp(-A(t))\exp((\tau-1)tD).$$

If $B(t, \tau)$ is the solution of the matrix differential equation

$$(*) \qquad \frac{\partial Y(t, \tau)}{\partial t} = \tau F(g(t, \tau), Y(t, \tau)), \qquad Y(0, \tau) = 0 \quad \textit{for each } \tau \in [0, 1]$$

and if $\|g(t, \tau)\| < \delta$ and $\|B(t, \tau)\| < \delta$ for each $t, \tau \in [0, 1]$, then

$$\varphi(t, \tau) = \exp(\tau t D)\exp(B(t, \tau)).$$

Proof of the lemma: Differentiation shows that

$$\varphi'(t)\varphi(t)^{-1} = D + \exp(tD)\left(\frac{d}{dt}\exp(A(t))\right)\exp(-A(t))\exp(-tD).$$

On the other hand, setting

$$\psi(t) := \exp(\tau t D)\exp(B(t, \tau))$$