ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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On the Convergence of the Multiplier Methods with Approximate Minimization

by Nguyen DINH HOA

I. Introduction. The convergence rate of multiplier methods with exact minimization of intermediate problems was investigated by Bertsekas [1] Tapia [2] and Rupp [3]. It was shown that the multiplier method using the Hestenes–Powell multiplier update formula or the projection formula is locally Q-linear convergent and that the multiplier method with Buys' update or Tapia's update is locally Q-quadratically convergent (For definition of the Q-convergence see [4]). The influence of nonexact minimization of intermediate problems on the convergence rate of the multiplier method of Hestenes and Powell was investigated by Rupp, Buys, Bertsekas. In [1] there was suggested a stopping criterion for the intermediate minimization such that the convergence rate is the same as in the method with exact minimization.

In this paper we deal with the convergence rate of the multiplier method using Buys' and projection updates (see (16) and (15) formulae, respectively) when the intermediate minimizations involved are approximate. We show (Proposition 3.1) that the multiplier methods are convergent when the minimizations are asymptotically approximate. The multiplier method using Buys' update is considered as the perturbation of Newton's method for dual problem. The conditions that guarantee that the method is locally Q-superlinear and Q-quadratically convergent are discussed.

II. The multiplier method. Consider the constrained optimization problem

$$\text{(1)} \qquad \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g(x) = 0\,, \end{cases}$$

where $f, g_1, \ldots, g_m$ are functionals on R^n and $g(x) = \big(g_1(x), \ldots, g_m(x)\big)^T$, with $m \leqslant n$. For $x \in R^n$ and $u \in R^m$ define

$$F(x, u) = f(x) + u^T \cdot g(x)\,.$$

The functional F is said to be *the Lagrangian functional* for problem (1). The vector $x \in R^n$ is said to be *a critical point* of problem (1) if there exists a Lagrange multiplier

$u \in R^m$ such that

(2) $$\nabla F(x, u) = \begin{pmatrix} \nabla f(x) + \nabla g(x) \cdot u \\ g(x) \end{pmatrix} = 0,$$

where $\nabla f(x)$ is the gradient of the functional $f(x)$ and $\nabla g(x) = (\nabla g_1(x), \ldots, \nabla g_m(x))$. Moreover, we say that x is *a nonsingular critical point* if the Hessian $\nabla^2 F(x, u)$ is invertible. If $x^* \in R^n$ is *a regular* (i.e. the gradients $\nabla g_1(x^*), \ldots, \nabla g_m(x^*)$ are linearly independent) *solution* of problem (1), then it is a critical point and the associated multiplier u^* is unique. Hence, an obvious approach to the constrained optimization (1) would be to apply the known methods for the nonlinear system of equations (2). However, such an approach is unfavourable, because the dimension of problem would be significantly increased and the Hessian $\nabla^2 F(x^*, u^*)$ is not positive definite.

Given a nonnegative number C, we consider the augmented (by penalty term) Lagrangian functional for problem (1):

(3) $$L(x, u, C) = f(x) + u^T \cdot g(x) + \frac{C}{2} g(x)^T \cdot g(x).$$

It was shown [5] that there exists $\bar{C} \geqslant 0$ such that for all $C \geqslant \bar{C}$ the following holds: if x^* is a nonsingular solution of problem (1) with associated Lagrange multiplier u^*, then x^* is a locally unique minimizer of $L(x, u^*, C)$ and $\nabla^2_{xx} L(x^*, u^*, C)$ is positive definite. Conversely, if x^* and u^* are such that x^* is a minimizer of $L(x, u^*, C)$ and $g(x^*) = 0$, then x^* is a solution of problem (1).

Let x^* be a nonsingular solution of problem (1) with associated Lagrange multiplier u^*, then $\nabla_x L(x^*, u^*, C) = 0$. Since

(4) $$\nabla^2_{xx} L(x^*, u^*, C) \quad \text{is positive definite},$$

by the implicit function theorem there exists a neighborhood W of u^* and a function $x(u)\colon W \subset R^m \to R^n$ such that the following holds:

(5) $$x(u^*) = x^*,$$

(6) $$\nabla_x L(x(u), u, C) = 0,$$

(7) $$\nabla x(u) = -\nabla_x g(x(u))^T \cdot \nabla^2_{xx} L(x(u), u, C)^{-1}.$$

For $u \in W$ define $h(u) = \min_x L(x, u, C)$. The function $h(u)$ is well defined on W and by (4), (5)

(8) $$h(u) = L(x(u), u, C)$$

Using (7) we obtain

(9) $$\nabla h(u) = g(x(u)) - \nabla x g(x(u))^T \cdot \nabla^2_{xx} L(x(u), u, C)^{-1} \cdot \nabla_x L(x(u), u, C),$$

(10) $$\nabla^2 h(u) = -\nabla_x g(x(u))^T \cdot \nabla^2_{xx} L(x(u), u, C)^{-1} \cdot \nabla_x g(x(u)) + \\ + \nabla^2 x(u) \cdot \nabla_x L(x(u), u, C)$$

but $\nabla_x L(x(u), u, C) = 0$ so that

$$\nabla h(u) = g(x(u)), \tag{11}$$

$$\nabla^2 h(u) = -\nabla_x g(x(u))^T \cdot \nabla_{xx}^2 L(x(u), u, C)^{-1} \cdot \nabla_x g(x(u)). \tag{12}$$

By (4) the Hessian $\nabla^2 h(u^*)$ is negative definite and the gradient $\nabla h(u^*) = g(x^*) = 0$. This proves that u^* solves the problem

$$\max_{u \in W} h(u). \tag{13}$$

Problem (13) is called *the dual problem* to the primal problem (1). We obtain the following Local Duality: if x^* solves the primal problem, then its associated Lagrange multiplier u^* solves the dual problem and x^* can be obtained from u^* as the solution of $\min_x L(x, u^*, C)$.

Hence, a natural approach to the constrained optimization problem is to solve the dual unconstrained problem (13) to obtain u^*. Although we do not know the explicit expression for the function $h(u)$, its gradient and Hessian are defined by (11) and (12), respectively. This is the idea of the multiplier method.

For the sake of simplicity, we assume that the penalty constant C remains fixed (though all the results obtained are true for C varying in a compact subsets of $[\bar{C}, \infty)$).

DEFINITION 2.1. A function $U: R^{n+m+1} \to R^m$ with the property that $u^* = U(x^*, u^*, C)$ whenever (x^*, u^*) is a critical point of problem (1) is said to be *a Lagrange multiplier update formula* for (3)

DEFINITION 2.2. (Multiplier Method) Let U be a Lagrange multiplier update formula. Then by *the multiplier method for problem* (1) *with* U we mean the following algorithm

Step 1. Determine a u^0, set $k_0 := 0$.

Step 2. Find $x(u^k)$ minimizing $L(x, u^k, C)$.

$$L(x(u^k), u^k, C) = \min_x L(x, u^k, C).$$

Step 3. $u^{k+1} = U(x(u^k), u^k, C)$.

Step 4. Go to Step 2 with $k := k+1$.

In practice a stopping criterion would be inserted in step 2 and also between step 2 and 3.

Consider the following multiplier update formulae

$$U_{HP}(x, u, C) = u + C \cdot g(x). \tag{14}$$

$$U_P(x, u, C) = -[\nabla g(x)^T \cdot \nabla g(x)]^{-1} \nabla g(x)^T \cdot \nabla f(x). \tag{15}$$

$$U_B(x, u, C) = u + [\nabla g(x)^T \cdot H \cdot \nabla g(x)]^{-1} \cdot g(x). \tag{16}$$

$$U_T(x, u, C) = [\nabla g(x)^T \cdot H \cdot \nabla g(x)]^{-1} \{g(x) - \nabla g(x)^T \cdot H \cdot (\nabla f(x) + C \cdot \nabla g(x) \cdot g(x))\}, \tag{17}$$

where $H := \nabla_{xx}^2 L(x, u, C)^{-1}$.

The multiplier update formula (14) obtained by using the gradient method for problem (13) was proposed independently by Hestenes and Powell. The formula (16) suggested by Buys is a result of the application of Newton's method to problem (13). The projection formula (15) is the solution for u of the system

$$(18)\qquad \begin{cases} \nabla_x L(x, u, C) = 0\,, \\ g(x) \qquad\quad\;\; = 0\,. \end{cases}$$

The formula (17) turns up when Newton's method is used for the system (18) and was suggested by Tapia. Having written the formula (17) in the form:

$$(19)\qquad U_T(x, u, C) = u + [\nabla g(x)^T \cdot H \cdot \nabla g(x)]^{-1}\{g(x) - \nabla g(x)^T \cdot H \cdot \nabla_x L(x, u, C)\}$$

we can consider the update $U_T(x, u, C)$ as a formula of Newton's type for problem (13) in which $\nabla h(u)$ and $\nabla^2 h(u)$ are given by (9) and (12) respectively.

It is interesting to observe that in the multiplier method the multiplier update formula (15) is equivalent to the Hestenes–Powell update (14), while Tapia's update (17) is equivalent to Buys' update (16), provided that in the step 2 minimization is exact.

The following result on the convergence rate of the multiplier method with exact minimization was established in [1], [2]

PROPOSITION 2.1. *Let x^* be a nonsingular solution of problem* (1) *with associated Lagrange multiplier u^*. Let the functions f, g_i be twice continuously differentiable in a neighborhood of x^*. Then,*

(i) *The multiplier method using the Hestenes–Powell update is locally Q-linearly convergent. Moreover, Q_1-factor is arbitrarily small if the penalty constant is sufficiently large.*

(ii) *The multiplier method using Buys' update is locally Q-quadratically convergent.*

We now establish a lemma which we will use in next Section.

LEMMA 2.2. *Let the assumptions of Proposition* 2.1 *hold. Then there exist a neighborhood of u^* and positive scalars $r_1, r_2 > 0$ such that in this neighborhood we have*

$$(20)\qquad r_1\|x(u') - x(u)\| \leqslant \|u' - u\| \leqslant r_2\|x(u') - x(u)\|\,,$$

where $x(u)$ is the implicit function defined by $\nabla_x L(x, u, C) = 0$.

Proof. By McLeod's mean value theorem [6] we have

$$(21)\qquad x(u') - x(u) = \sum_{i=1}^{m} t_i \cdot \nabla x\big(u + \theta_i(u' - u)\big) \cdot (u' - u)\,,$$

where $0 < \theta_i < 1$, $t_i \geqslant 0$, $\sum_{i=1}^{m} t_i = 1$. By (4) and Lemma on perturbation ([4], p. 46) there exists a neighborhood W_1 of u^* in which the inverse Hessian $\nabla_{xx}^2 L(x(u), u, C)^{-1}$ is bounded uniformly. Hence, from (7) we have $\|\nabla x(u)\| \leqslant M$ in this neighborhood. This proves the existence of r_1.

Let us use the notation $u^i = u+\theta_i(u'-u)$. From (21) and (7) we obtain

$$\nabla_x g(x(u))^T\cdot(x(u')-x(u)) = \sum_{i=1}^m t_i\cdot\nabla_x g(x(u))^T\cdot\nabla x(u^i)\cdot(u'-u)$$
$$= -\sum_{i=1}^m t_i\cdot\nabla_x g(x(u))^T\cdot\nabla_{xx}^2 L(x(u^i), u^i, C)\cdot\nabla_x g(x(u^i))\cdot(u'-u)\,.$$

By (4) there exists a neighborhood W_2 of u^* in which the matrix

$$\sum_{i=1}^m t_i\cdot\nabla_x g(x(u))^T\cdot\nabla_{xx}^2 L(x(u^i), u^i, C)\cdot\nabla_x g(x(u^i))$$

is invertible and its inverse is bounded uniformly. This establishes the existence of r_2. So (20) is true in $W_1\cap W_2$.

Setting $u = u^*$, $u' = u^k$ in (20) we obtain

$$r_1\|x(u^k)-x^*\|\leqslant\|u^k-u^*\|\leqslant r_2\|x(u^k)-x^*\|\,.$$

This estimation proves the following

COROLLARY 2.3. (Proposition 9.1, [2]) *Suppose that the multiplier method with an arbitrary multiplier update formula generates the sequence* $\{(x(u^k), u^k)\}$. *Then* $u^k\to u^*$ *with Q-order q if and only if* $x(u^k)\to x^*$ *with Q-order q.*

III. Convergence of the multiplier with approximate minimization. Remember that in step 2 of the multiplier method we need to find $x(u^k)$ such that $L(x(u^k), u^k, C) = \min_x L(x, u^k, C)$. In practice, generally, it is impossible to calculate $x(u^k)$ exactly. Usually, a stopping criterion, for instance, $\|\nabla_x L(x^k, u^k, C)\|\leqslant\varepsilon_k$ would be applied and the approximate minimizer x^k is used in the calculation of u^{k+1}.

Now we will prove that the multiplier method is convergent with asymptotically exact approximate minimization.

PROPOSITION 3.1. *Let $U(x, u, C)$ be a multiplier update formula for problem* (1). *Assume that U is continuous in a neighbourhood of (x^*, u^*) and the multiplier method with $U(x, u, C)$ is locally Q-linearly convergent. Then there exist $\delta_1, \delta_2>0$ such that the sequence generated by $u^{k+1} = U(x^k, u^k, C)$ converges to u^*, provided that:*

$$\begin{cases}\|u^0-u^*\|\leqslant\delta_1\,,\\ \|x^k-x(u^k)\|\leqslant\delta_2 \text{ for all } k, \text{ and } \|x^k-x(u^k)\|\to 0\,.\end{cases}\tag{22}$$

Proof. By the assumption the multiplier method with U is locally Q-linearly convergent, so we can assume, without loss of generality, that there exists a neighborhood N of u^* such that if $u^0\in N$ then $\tilde u^{k+1} = U(x(\tilde u^k), \tilde u^k, C)$ remains in this neighborhood and

$$\frac{\|\tilde u^{k+1}-u^*\|}{\|\tilde u^k-u^*\|}\leqslant\alpha<1 \qquad \text{for all } k\,.\tag{23}$$

Take a sufficiently small δ_1, ε such that the closed ball $B\left(u^*, \delta_1 + \frac{\varepsilon}{1-\alpha}\right)$ is contained in the neighborhood N. By the continuity of U there exists a neighborhood of x^* and a positive scalar δ_2 such that in this neighborhood if $\|x'-x\| \leqslant \delta_2$ then $|U(x', u, C) - - U(x, u, C)|| \leqslant \varepsilon$ for all $u \in B$.

Let u^0 be an initial approximation of the multiplier, $\|u^0 - u^*\| \leqslant \delta_1$ and $u^{k+1} = U(x^k, u^k, C)$, where $\|x^k - x(u^k)\| \leqslant \frac{\delta}{2}$. We have the following estimation:

$$\begin{aligned} &\|u^{k+1}-u^*\| = \|U(x^k, u^k, C) - u^*\|\,, \\ &\|u^{k+1}-u^*\| \leqslant \|U(x(u^k), u^k, C) - u^*\| + \|U(x^k, u^k, C) - U(x(u^k), u^k, C)\|\,. \end{aligned} \tag{24}$$

Hence, using (23) we obtain

$$\|u^{k+1}-u^*\| \leqslant \alpha \|u^k - u^*\| + \varepsilon \leqslant \alpha^{k+1} \|u^0 - u^*\| + \frac{\varepsilon}{1-\alpha}\,.$$

This proves that $u^k \in B\left(u^*, \delta_1 + \frac{\varepsilon}{1-\alpha}\right)$ for all k. S the sequence $\{u^k\}$ is bounded. We will now prove that the sequence $\{u^k\}$ has the unique limit point u^* if $\|x^k - x(u^k)\| \to 0$.

Let $\{u^{(l)}\}$ be a subsequence of $\{u^k\}$ which is convergent to $\bar{u}$. If $\bar{u} \neq u^*$ we denote the distance between these points by d. Take $\delta > 0$ such that $(d-\delta) - \alpha(d+\delta) > 0$. There exists L such that for all $l > L$ we have $\|u^{(l)} - \bar{u}\| < \delta$ and $\varepsilon_l = \|U(x(u^{(l)}), u^{(l)}, C) - U(x^{(l)}, u^{(l)}, C)\| < (d-\delta) - \alpha(d+\delta)$. But

$$\|u^{(l)+1} - u^*\| \leqslant \alpha \|u^{(l)} - u^*\| + \varepsilon_l \leqslant \alpha(d+\delta) + \varepsilon_l < d - \delta\,.$$

This proves that $u^{(l)+1} \notin B(\bar{u}, \delta)$. In the analogous way we conclude that $u^{(l)+s} \notin B(\bar{u}, \delta)$ for all s. This contradicts the assumption that $u^{(l)} \to \bar{u}$. Thus $d = 0$ and proposition is established.

The inequality (24) shows that the good convergence behaviour of the multiplier method may be destroyed if $\|x^k - x(u^k)\|$ do not decreases to zero sufficiently fast. Analising the convergence of the Hestenes–Powell multiplier Bertsekas [1] has given a numerical example in which the method with nonexact minimization generates the sequence $\{u^k\}$ which converges to u^* but the convergence is not Q-linear. He proposed a special stopping criterion for minimization with which the method has the same rate of convergence as the method with exact minimization. Namely, we have

PROPOSITION 3.2. [1] *Assume that function $g(x)$ is Lipschitz continuous in a neighborhood of x^*: $\|g(x) - g(y)\| \leqslant L\|x - y\|$. Then the Hestenes–Powell multiplier method with nonexact minimization has the same rate of convergence as the method with exact minimization, provided that*

$$\|\nabla_x L(x^k, u^k, C)\| \leqslant \eta_k \|g(x^k)\|\,, \quad \textit{and} \quad \eta_k \to 0\,. \tag{25}$$

For the convenience of the reader we summarize the proof of Bertsekas. Next we will use some estimations obtained here.

Proof. Since $V_{xx}^2 L(x^*, u^*, C)$ is positive definite there exists a neighborhood $\mathcal{N}$ of (x^*, u^*) such that for all $(x, u) \in \mathcal{N}$, $\nabla_{xx}^2 L(x, u, C)$ is invertible and its inverse is bounded uniformly, ([4], Lemma on perturbation), $\|\nabla_{xx}^2 L(x, u, C)\| \leqslant M$. Take a sufficiently large k_0 such that (x^k, u^k), $(x(u^k), u^k) \in \mathcal{N}$ for all $k \geqslant k_0$. By McLeod's mean value theorem we have

$$\nabla_x L(x^k, u^k, C) - \nabla_x L(x(u^k), u^k, C)$$
$$= \sum_{i=1}^{n} t_{k,i} \nabla_{xx}^2 L\big(x(u^k) + \theta_{k,i}(x^k - x(u^k)), u^k, C\big) \cdot \big(x^k - x(u^k)\big),$$

where $0 < \theta_{k,i} < 1$, $t_{k,i} \geqslant 0$, $\sum_{i=1}^{m} t_{k,i} = 1$.

Hence, we obtain $\|x^k - x(u^k)\| \leqslant M \|\nabla_x L(x^k, u^k, C)\|$ and by (25)

$$\begin{aligned} \|x^k - x(u^k)\| &\leqslant M\eta_k \|g(x^k)\| \\ &\leqslant M\eta_k\big(\|g(x^k) - g(x(u^k))\| + \|g(x(u^k)) - g(x^*)\|\big) \\ &\leqslant ML\eta_k\big(\|x^k - x(u)^k\| + \|x(u^k) - x^*\|\big). \end{aligned}$$

Using (20) we get

$$|x^k - x(u^k)\| \leqslant \frac{ML\eta_k}{1 - ML\eta_k} \|x(u^k) - x^*\| \leqslant \frac{ML\eta_k}{1 - ML\eta_k} \frac{1}{r_1} \|u^k - u^*\| .$$

Setting $\sigma(\eta_k) = \dfrac{ML\eta_k}{1 - ML\eta_k} \cdot \dfrac{1}{r_1}$ we have

(26) $$\|x^k - x(u^k)\| \leqslant \sigma(\eta_k) \|u^k - u^*\| , \quad \text{where } \sigma(\eta_k) \to 0 \quad \text{if } \eta_k \to 0 .$$

It is well known [1], [2] that the sequence $\{v^k\}$ generated by the Hestenes–Powell method Q-linearly converges to u^*, i.e. given

$$v^{k+1} = v^k + C \cdot g(x(v^k)) \text{ we have } \overline{\lim_{k \to \infty}} \frac{\|v^{k+1} - u^*\|}{\|v^k - u^*\|} = q < 1 .$$

Hence, for an arbitrary small ε there exists δ such that if $\|v^k - u^*\| < \delta$, then $\|v^k + C \cdot g(x(v^k)) - u^*\| \leqslant (q + \varepsilon)\|v^k - u^*\|$. When the approximation x^k is taken instead of $x(v^k)$, The Hestenes–Powell multiplier method generates the sequence $\{u^k\}$ by the iteration

$$u^{k+1} = u^k + C \cdot g(x^k) .$$

By Proposition 3.1 and (25) the sequence $\{u^k\}$ is convergent to u^*. Assume that k_0 is large enough in order that $\|u^k - u^*\| < \delta$ for all $k \geqslant k_0$. By using (26) we estimate

$$\begin{aligned} \|u^{k+1} - u^*\| &= \|u^k + C \cdot g(x^k) - u^*\| \\ &\leqslant \|u^k + C \cdot g(x(u^k)) - u^*\| + C\|g(x^k) - g(x(u^k))\| \\ &\leqslant (q + \varepsilon)\|u^k - u^*\| + CL\|x^k - x(u^k)\| \end{aligned}$$

and, therefore,

(27) $$\|u^{k+1} - u^*\| \leqslant (q + \varepsilon + CL\sigma(\eta_k))\|u^k - u^*\| .$$

Since ε is arbitrarily small and $\sigma(\eta_k) \to 0$, the inequality (27) proves the proposition.