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DCLXI UNIVERSITATIS IACELLONICAE ACTA MATHEMATICA, FASCICULUS XXIV 1984

Difference Inequalities for the Equation of Diffusion

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§ 1. In this note we would like to indicate that the convergence of the difference method for the heat equation can be treated with the aid of difference inequalities.

The beauty and simplicity of the method of difference inequalities are the cause I include it with pleasure into the lecture on partial differential equations.

Let us consider the boundary value problem for the function $u(x, t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \cdot \dfrac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = \varphi(x), \\ u(0, t) = \psi_1(t),\ u(l, t) = \psi_2(t), \end{cases} \tag{1.1}$$

and the corresponding difference problem for the discrete function v_{ij}:

$$\begin{cases} \dfrac{v_{i,j+1}-v_{ij}}{k} = c^2 \cdot \dfrac{v_{i+1,j}-2v_{ij}+v_{i-1,j}}{h^2}, \\ v_{i,0} = \varphi(x_i), \\ v_{0,j} = \psi_1(t_j),\ v_{N,j} = \psi_2(t_j), \end{cases} \tag{1.2}$$

both problems being considered in the domain

$$D\colon 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

$$\begin{cases} x_i = i \cdot h,\ t_j = j \cdot k,\ 0 < h = \dfrac{l}{N}, \\ 0 < k = \dfrac{T}{N_1}, \quad i = 0, 1, \ldots, N; \quad j = 0, 1, \ldots, N_1. \end{cases} \tag{1.4}$$

Let us denote by u_{ij} the value of the solution $u(x, t)$ at the nodal point (x_i, t_j). Obviously, we have

$$\frac{u_{i,j+1}-u_{ij}}{k} = c^2 \cdot \frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2} + \varepsilon_{ij}(h, k), \tag{1.5}$$

since u_{ij} does not satisfy, in general, the difference equation. In addition, the error $\varepsilon(h, k)$ defined by

$$(1.6) \qquad \varepsilon(h, k) = \max_{i,j} |\varepsilon_{ij}(h, k)| ,$$

satisfies the relation

$$(1.7) \qquad \varepsilon(h, k) \to 0 , \quad \text{as } h \to 0,\ k \to 0 ,$$

for the solution $u(x, t)$ of the class C^2 in the domain D.

§ **2.** Let us introduce the error r_{ij}:

$$(2.1) \qquad r_{ij} = u_{ij} - v_{ij} ,$$

and the maximal value s_j on the fixed time level t_j:

$$(2.2) \qquad s_j = \max_i r_{ij} .$$

The maximal values are attained at some nodal points:

$$(2.3) \qquad s_{j+1} = r_{\alpha,j+1} , \quad s_j = r_{\beta j} ,$$

therefore we can write

$$(2.4) \qquad \frac{s_{j+1} - s_j}{k} = \frac{r_{\alpha,j+1} - r_{\beta j}}{k} ,$$

and

$$(2.5) \qquad \frac{s_{j+1} - s_j}{k} = \frac{r_{\alpha,j+1} - r_{\alpha j}}{k} + \frac{r_{\alpha j} - r_{\beta j}}{k} .$$

Let us suppose for instance that the nodal point (x_α, t_{j+1}) is in the interior of the domain D. Then the first member on the right-hand side of (2.5) can be obtained immediately. It is sufficient only to subtract (1.5) and the first relation (1.2):

$$(2.6) \qquad \frac{r_{\alpha,j+1} - r_{\alpha j}}{k} = c^2 \cdot \frac{r_{\alpha+1,j} - 2r_{\alpha j} + r_{\alpha-1,j}}{h^2} + \varepsilon_{\alpha j}(h, k) .$$

Taking into account (2.5) and (2.6) we obtain

$$(2.7) \qquad \frac{s_{j+1} - s_j}{k} = c^2 \cdot \frac{r_{\alpha+1,j} - 2 \cdot r_{\alpha j} + r_{\alpha-1,j}}{h^2} + \varepsilon_{\alpha j} + \frac{r_{\alpha j} - r_{\beta j}}{k} .$$

Let us introduce now the maximal value $r_{\beta j}$ at appropriate places in the formula (2.7):

$$(2.8) \qquad \frac{s_{j+1} - s_j}{k} = c^2 \cdot \frac{(r_{\alpha+1,j} - r_{\beta j}) - 2 \cdot (r_{\alpha j} - r_{\beta j}) + (r_{\alpha-1,j} - r_{\beta j})}{h^2} + \varepsilon_{\alpha j} + \frac{r_{\alpha j} - r_{\beta j}}{k} .$$

Collecting terms we get

$$\frac{s_{j+1}-s_j}{k} = \varepsilon_{\alpha j}+\frac{c^2}{h^2}\cdot(r_{\alpha+1,j}-r_{\beta j})+\frac{c^2}{h^2}\cdot(r_{\alpha-1,j}-r_{\beta j})+\left(\frac{1}{k}-\frac{2c^2}{h^2}\right)\cdot(r_{\alpha j}-r_{\beta j}). \tag{2.9}$$

Let us assume that

$$\frac{1}{k}-\frac{2c^2}{h^2}\geqslant 0. \tag{2.10}$$

Then, from the definition of the maximal value $r_{\beta j}$ it follows that

$$\begin{cases} r_{\alpha+1,j}-r_{\beta j}\leqslant 0,\ r_{\alpha-1,j}-r_{\beta j}\leqslant 0, \\ r_{\alpha j}-r_{\beta j}\leqslant 0,\ \left(\frac{1}{k}-\frac{2c^2}{h^2}\right)(r_{\alpha j}-r_{\beta j})\leqslant 0. \end{cases} \tag{2.11}$$

Thus, from (2.9) and (2.11) we get the desired difference inequality and the initial condition:

$$\frac{s_{j+1}-s_j}{k}\leqslant\varepsilon(h,k), \quad \varepsilon_0 = 0. \tag{2.12}$$

The inequality (2.12) remains valid if the nodal point (x_α, t_{j+1}) is on the boundary i.e. $\alpha = 0$ or $\alpha = N$. In fact, we have in this case $s_{j+1} = r_{\alpha,j+1} = 0$ and the left-hand side of (2.12) reduces to the non-positive quantity $-s_j/k$ whereas the right-hand side of (2.12) is non-negative.

§ **3.** The difference inequality (2.12) can be solved immediately. First we write (2.12) in the form

$$s_{j+1}\leqslant s_j+k\cdot\varepsilon(h,k). \tag{3.1}$$

From (3.1) we obtain successively

$$\begin{cases} s_0 = 0, \\ s_1 \leqslant 0+k\cdot\varepsilon, \\ s_2 \leqslant s_1+k\cdot\varepsilon\leqslant 2\cdot k\cdot\varepsilon, \\ s_3 \leqslant s_2+k\cdot\varepsilon \leqslant 3\cdot k\cdot\varepsilon, \\ \dots\ \dots\ \dots\ \dots \end{cases} \tag{3.2}$$

and by induction:

$$s_j\leqslant j\cdot k\cdot\varepsilon(h,k) \quad (j = 0, 1, \dots, N_1) \tag{3.3}$$

But

$$0\leqslant j\cdot k\leqslant N_1\cdot k = N_1\cdot\frac{T}{N_1} = T, \tag{3.4}$$

therefore we have

(3.5) $$0 \leqslant j \cdot k \leqslant T, \quad \text{for } j = 0, 1, \ldots, N_1; \quad k = \frac{T}{N_1},$$

and the solution of the difference inequality (2.12) has the form

(3.6) $$s_j \leqslant T \cdot \varepsilon(h, k) \quad (j = 0, 1, \ldots, N_1).$$

§ **4.** In the similar way let us introduce the minimal value

(4.1) $$z_j = \min_i r_{ij}$$

Then we have

(4.2) $$z_{j+1} = r_{\gamma, j+1}, \quad z_j = r_{\delta_j},$$

at appropriate nodal points.

We can write

(4.3) $$\frac{z_{j+1} - z_j}{k} = \frac{r_{\gamma, j+1} - r_{\gamma j}}{k} + \frac{r_{\gamma j} - r_{\delta j}}{k},$$

and repeat the similar reasoning as in § 2 which leads us to the difference inequality

(4.4) $$\frac{z_{j+1} - z_j}{k} \geqslant -\varepsilon(h, k), \quad z_0 = 0.$$

The difference inequality (4.4) can be solved as in § 3 and we get

(4.5) $$z_j \geqslant -T \cdot \varepsilon(h, k) \quad (j = 0, 1, \ldots, N_1).$$

§ 5. Taking into account (3.6) and (4.5) we obtain the inequalities for the errr r_{ij}:

(5.1) $$-T \cdot \varepsilon(h, k) \leqslant z_j \leqslant r_{ij} \leqslant s_j \leqslant T \cdot \varepsilon(h, k),$$

which means that the following error estimate

(5.2) $$|r_{ij}| \leqslant T \cdot \varepsilon(h, k),$$

holds for $i = 0, 1, \ldots, N$; $j = 0, 1, \ldots, N_1$.

Since the error $\varepsilon(h, k)$ approaches zero as $h \to 0$, $k \to 0$, cf. (1.7), we see that

(5.3) $$r_{ij} \to 0, \quad \text{as } h \to 0, \ k \to 0,$$

which means that the difference method is convergent.

From our discussion of difference inequalities it follows that the difference method is convergent and the error estimate (5.2) holds.

This ends our considerations.

Received December 30, 1980.