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Two Criteria for Continuity of Polynomials and G-holomorphic Mappings in Infinite Dimensions

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Abstract. The following two theorems are proved. THEOREM 1. *If p is a polynomial from a complex topological vector space into the field $\boldsymbol{C}$ and $p^{-1}(0)$ is closed, then p is continuous.* THEOREM 2. *If f is a G-holomorphic mapping* (resp. *real polynomial mapping*) *from a complex* (resp. *real*) *Baire space into a semi-polinormed space and the graph of f is closed, then f is continuous.*

Introduction. All topological vector spaces (t.v.s.) in this note are required to be complex or real Hausdorff spaces. A homogeneous polynomial h of degree k from a t.v.s. E to a t.v.s. F is the restriction of k-linear symmetrical mapping $\hat{h}: E^k \to F$ to the diagonal i.e., $h(x) = \hat{h}(x, \ldots, x)$. We write $h \in Q^k(E, F)$, $Q^0(E, F) = F$. A polynomial p is a finite sum of homogeneous polynomials and we write $p \in Q(E, F)$. This definition is equivalent to the following: $p: E \to F$ is a polynomial of degree at most n if p is a polynomial of degree at most n on every affine line contained in E (cf. [1], [4], [6]). The above suggests the following: A function $f: U \to F$, where U is an open subset of a t.v.s. E over $\boldsymbol{C}$ is said to be G-holomorphic if for every affine line L f restricted to $U \cap L$ is holomorphic ([2], [4]). A graph of a function f with domain $D \subset E$ is the set $G(f) := \{(x, f(x)) \in E \times F: x \in D\}$. We begin with the following

THEOREM 1. *If E is a complex* t.v.s., *$p \in Q(E, \boldsymbol{C})$ and* $\ker p := p^{-1}(0)$ *is closed or non-dense in E, then p is continuous.* (I suppose that the theorem 1 is probably known but I have not seen its proof anywhere.)

Theorem 1 is not true if E is a t.v.s. over $\boldsymbol{R}$, $\dim E = \infty$, even when p is a homogeneous polynomial.

Example 1. $E := \boldsymbol{R}^{(N)}$ endowed with the topology from l^2 is a unitary space. We put $h(x) = \sum_{k=1}^{\infty} k^2 x_k^2$, $x = (x_1, x_2, \ldots) \in E$; $h \in Q^2(E, \boldsymbol{R})$, $\ker h = \{0\}$, h is evidently not continuous.

The following fact for linear mappings:
If E is a t.v.s. over the field $K (K = \boldsymbol{C}$ or $K = \boldsymbol{R})$, $p \in Q^1(E, K^n)$ and $\ker p$ is closed, then

p is continuous; suggests the question: will the above theorem be in force when we take $K = C$ and $p \in Q(E, C^n)$? The answer is negative even if $p \in Q^2(E, C^2)$ provided $\dim E = \infty$.

Example 2. If f is continuous and h is not continuous linear form in E, then we put $p(x) := (f^2(x), f(x)h(x))$, $x \in E$ and we see that $\ker p = \ker f$ is closed but p is not continuous.

Question: Is a surjective polynomial with closed kernel continuous in the above case? And if additionally the graph of p is closed, is then p continuous?

It is understood that we would like to have a theorem of the "Closed Graph Theorem" type for polynomials and G-holomorphic functions.

A Montel space is a locally convex space (l.c.s.) in which every closed bounded set is compact (cf. [5]). Let X be a Montel space and let $\{q_s : s \in S\}$ denote a filtrant family of seminorms determining the topology on X. Let n be an increasing positive valued net i.e., $n: S \to (0, \infty)$ and $n(s) > n(r)$ if $s \succ r$, where $(S, \succ)$ is a direct set. Let $F(n)$ be a subspace of all elements y of X for which there exists $C(y) > 0$ such that $q_s(y) \leqslant C(y)\, n(s)$ for every $s \in S$. Let J be a countable direct set of the mentioned nets n i.e., for nets $n, \bar{n} \in J$ we write $n \succ \bar{n}$ if $n(s) > \bar{n}(s)$ for each $s \in S$. We set $F := \bigcup_{n \in J} F(n)$. Then F is a t.v.s. endowed with the relative topology from X. We call F the semi-polynormed space (cf. [3]).

Remark. The class of all semi-polynormed spaces is nontrivial in the sense that any space F need not be a finite dimensional vector space. For example F may be the space of all holomorphic functions that are δ-tempered increasing ([3]), e.g., let D be an open set in C^n and let X be the space of all holomorphic functions in D endowed with the topology given by the family of seminorms $q_s(f) := \sup\left\{|f(z)| : \operatorname{dist}(z, \setminus D) \leqslant \frac{1}{s}\right\}$, $s \in N$. We set $F(n) := \{f \in X : q_s(f) \leqslant C(f) s^n,\ s \in N\}$ and $F := \bigcup_{n \in N} F(n)$.

Now we can prove the following

THEOREM 2. *If U is an open subset of a complex* (resp. *real*) *Baire* t.v.s. *E, f is a G-holomorphic function from U into a semi-polynormed space F* (resp. *f is a real polynomial*) *and the graph of f is closed, then f is continuous.*

Proof of Theorem 1. (i) p is a nontrivial homogeneous polynomial of degree n. For some fixed $a \in E \setminus \ker p$ there exists a balanced neighbourhood V of 0 in E such that $a + V \subset E \setminus \ker p$ ([5]).

Consider the equation

$$p(a+tv) = \sum_{k=0}^{n} t^k \binom{n}{k} \hat{p}(v, \ldots, v, a, \ldots, a) = 0, \quad v \in V, \quad t \in C.$$

For each fixed $v \in V$ such that $p(v) \neq 0$ the polynomial $p(a+tv)$ has exactly n zeros $t_1, \ldots, t_n$ (they are counted with their multiplicities). By Viete's formula we have

$$|p(v)t_1 \ldots t_n| = |p(a)|.$$

Since $(a+V) \cap \ker p = \emptyset$, so $|t_k| > 1$, $k = 1, \ldots, n$, and consequently $|p(v)| < |p(a)|$ for $v \in V$. Since p is bounded in V, p is continuous ([1]).

(ii) p is a nontrivial polynomial of degree n. Similarly as above we put

$$p(a+tv) = \sum_{k=0}^{n} h_k(a+tv) = 0, \quad a+V \subset E \setminus \ker p,\ t \notin C,\ h_k \in Q^k(E, C).$$

By trivial calculations we obtain

$$t^n h_n(v) + \ldots + t^{n-k} \sum_{j=n-k}^{n} \binom{j}{n-k} \hat{h}_j(v, \ldots, v, a, \ldots, a) + \ldots + \sum_{j=0}^{n} h_j(a) = 0. \qquad (*)$$

Since $(a+V) \cap \ker p = \emptyset$, so $|t_k| > 1$, where t_k is a root of the equation $(*)$, $k = 1, \ldots, n$. Analogically as before we get $|h_n(v)| < |A|$ for $v \in V$, where A is the last term of the left-hand side of $(*)$. Since h_n is continuous, the generating n-linear symmetric form $\hat{h}_n$ is also continuous ([6]). Assuming the continuity of $\hat{h}_n, \hat{h}_{n-2}, \ldots, \hat{h}_{n-k+1}$ we will check the continuity of h_{n-k}. By Viéte's formulas, if $h_n(v) \neq 0$ we have

$$h_n(v) \sum t_{a(1)} \cdots t_{a(k)} = (-1)^k \left\{ h_{n-k}(v) + \sum_{j=n-k+1}^{n} \binom{j}{n-k} h_j(v, \ldots, v, a, \ldots, a) \right\}, \qquad (**)$$

where the sum in the left-hand side of $(**)$ is spread over all k-elements combinations of n-elements.

The left-hand side of $(**)$ is equal to

$$h_n(v) \sum \frac{t_1 \ldots t_n}{t_{b(1)} \cdots t_{b(n-k)}},$$

where the last sum is taken over all $(n-k)$-elements combinations on n elements. Since $|t_j| > 1$, so the absolute value of the left hand side of $(**)$ is less than $\binom{n}{k}|A|$ (this is because $h_n(v) \cdot t_1 \cdot \ldots \cdot t_n = A$). Consequently, owing to the induction conjecture we derive from $(**)$ the boundedness of h_{n-k} in V. Q.E.D.

Proof of Theorem 2. We may suppose without loss of generality that U is connected and let us put

$$E(k, n) := \{x \in U \colon q_s \circ f(x) \leqslant kn(s) \quad \text{for every } s \in S\}, \quad n \in J,\ k = 1, 2, \ldots$$

We will check that $E(k, n)$ is closed in U. We take a net $\{x_a\} \subset E(k, n)$ which converges to a point $x \in U$. By Montel property of X the set $F(k, n) := \{y \in F(n) \colon q_s(y) \leqslant kn(s)$ for every $s \in S\}$ is compact in $F(n)$ and hence there exists a subnet $\{f(x_b)\}$ of the net

$\{f(x_a)\} \subset F(k, n)$ such that $\{f(x_b)\}$ converges to $y \in F(k, n)$. Therefore $f(x) = y$ and $x \in E(k, n)$ because $G(f)$ is closed. So any $E(k, n)$ is closed and $\bigcup_{(k,n) \in N \times J} E(k, n) = U$ because $\bigcup_{k,n} F(k, n) = F$. From the Baire property of U we obtain $\operatorname{int} E(k_0, n_0) \neq \emptyset$. Since f is bounded in a non-void open subset of U and E is a Baire t.s.v., f is continuous ([2]). Q.E.D.

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