ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DCLXI UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXIV 1984

Polynomial Automorphisms of C^n

by Kamil RUSEK and Tadeusz WINIARSKI

Abstract. The present paper consists of two parts. In the first part we give a sharp estimation of the degree of the inverse to a polynomial automorphism of C^n. The second part contains the theorem on limits of sequences of polynomial automorphisms with some of its applications.

Introduction. In Part One we give a sharp estimation of the degree of the inverse to a polynomial authomorphism of C^n. (Theorem 1.5). This result is based on the affine version of Bezout's theorem. Since we can not find in bibliography such a version of Bezout's theorem we present its proof based on Rouche's theorem and on the well known projective version of Bezout's theorem (cf. [6], p. 191).

In Part Two a structure of some sets of polynomial automorphisms is examined. In particular, a formal analogue of known Cartan's theorem on sequences of biholomorphisms of a bounded domain (cf. [3], p. 78) is proved (Theorem 2.3). The constructibility and algebraicity of the sets of polynomial automorphisms of bounded degree and fixed jacobian are also given (Theorems 2.5 and 2.6).

We introduce some notations and definitions which will be frequently used in the sequel.

Let M, N be complex vector spaces. Let $F\colon M \to N$ be a polynomial mapping and let $F = \sum_0^\infty F_k$ be the decomposition of F into the homogeneous components. As usual, the number $\deg F = \max\{k\colon F_k \neq 0\}$ is called the degree of F. Let us write, for $k \in \mathbf{N}$,

$$\mathscr{P}^k(M, N) = \{F\colon M \to N,\ F \text{ is a polynomial and } \deg F \leqslant k\},$$

$$\mathscr{P}^k(M) = \mathscr{P}^k(M, M),\ C^k[M] = \mathscr{P}^k(M, C),$$

$$\mathscr{P}(M, N) = \bigcup_1^\infty \mathscr{P}^k(M, N),\ \mathscr{P}(M) = \mathscr{P}(M, M),$$

$$C[M] = \mathscr{P}(M, C).$$

DEFINITION. We call a mapping $F\colon M \to M$ a *polynomial automorphism* of the space M, if F is one-to-one polynomial transformation of M onto M and the mapping F^{-1} is polynomial too.

We write

$$\mathscr{P}^k_A(M) = \{F \in \mathscr{P}^k(M):\ F \text{ is a polynomial automorphism of } M\},$$

$$\mathscr{P}_A(M) = \bigcup_1^{\infty} \mathscr{P}^k_A(M).$$

Assume that M is a finite-dimensional space.

The mapping J: $\mathscr{P}(M) \ni F \to \det F' \in \boldsymbol{C}[M]$ is a polynomial mapping. Thus for $\lambda \in \boldsymbol{C}$, $k \in \mathbf{N}$, the set

$$\mathscr{P}^k_\lambda(M) = \{F \in \mathscr{P}^k(M):\ J(F) = \lambda\}$$

is algebraic in the vector space $\mathscr{P}^k(M)$.

We shall need also some notations from the projective geometry.

If M is a finite dimensional complex vector space, we denote by $\boldsymbol{P}(M)$ the respective projective space, i.e. the space of vector lines through the origin in M. Define φ: $M \ni z \to \boldsymbol{P}((1, z)) = \boldsymbol{C}(1, z) \in \boldsymbol{P}(\boldsymbol{C} \times M)$. For an arbitrary set $S \subset M$ the set $\varphi(S)$ will be denoted by $\boldsymbol{P}(S)$.

Part One. In the beginning we present an affine version of Bezout's theorem.

Proposition 1.1. *Let* $\dim M = n \geqslant 2$ *and* $F = (f_1, \dots, f_n)$: $M \to \boldsymbol{C}^n$ *be a polynomial mapping such that* $F^{-1}(0) = \{a_1, \dots, a_k\}$. *Then*

$$\nu(F) = \sum_{i=1}^{k} m_{a_i} F \leqslant \deg f_1 \dots \deg f_n,$$

where $m_{a_i}F$ *denotes the multiplicity of* F *at the point* a_i (see e.g. Stoll [7] for the definition).

The proof is preceded by two lemmas which permit us to use the projective version of Bezout's theorem

Lemma 1.2. *Let* V_1, V_2 *be homogeneous algebraic sets in* M *of pure dimensions* k, l, *respectively* $(1 \leqslant k \leqslant l < n)$. *Then there exists a linear isomorphism* σ *of* M *such that*

$$\dim(V_1 \cap \sigma(V_2)) < k.$$

Proof. Let $V_1 = V_1^1 \cup \dots \cup V_1^s$ be the decomposition of V_1 into its irreducible components (which are also homogeneous). Let us fix a point a_j in $\boldsymbol{P}(V_1^j)$ for $j = 1, \dots, s$, and a hyperplane X in M such that $X \cap a_j = 0$, $j = 1, \dots, s$. Let $Y \subset M$ be a vector line, complementary to X such that $Y \cap V_2 = \{0\}$. Fix in turn a point $y_0 \in Y \setminus \{0\}$ and two constants r, $R > 0$ such that

$$(a_1 \cup \dots \cup a_s) \cap (y_0 + X) \subset y_0 + B_X(R),$$

$$y_0 + B_X(r) \subset (y_0 + X) \setminus V_2$$