

Some Remarks on Processes II

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In the papers [1], [2] C. Dafermos gives a theory of processes. The process has a meaning for the differential equation $x' = f(t, x)$, similar to that of the dynamical system for the differential equation $x' = f(x)$. Dafermos obtained his most important results for compact processes. In this paper we show that in some sense the theory of compact processes is adequate the theory of the differential equations of the form $x' = f(t, x)$ with f almost periodic.

We recall a pseudoprocesses definition (see [4], [5]). Let X be an abstract space, $(W, +)$ an abelian group, $U \subset W$ a subsemigroup such that $0 \in U$. Let $u: W \times X \times U \rightarrow X$ be a mapping.

We call the system (X, W, U, u) a *pseudoprocess* if

- (a) $u^t(x, 0) = x$ for all $t \in W$, $x \in X$;
- (b) $u^t(x, s + \tau) = u^{t+s}(u^s(x, s), \tau)$ for all $t \in W$, $x \in X$, $\tau, s \in U$.

We consider a differential equation

$$(1) \quad x' = f(t, x),$$

where $f: R \times R^n \rightarrow R^n$ is a continuous function such that for every $(t_0, x_0) \in R \times R^n$ there exists exactly one solution $\varphi_{(t_0, x_0)}: [t_0, \infty) \rightarrow R^n$ of equation (1) with the initial condition $x(t_0) = x_0$.

If we put

$$u^{t_0}(x_0, \tau) = \varphi_{(t_0, x_0)}(t_0 + \tau)$$

then we obtain a pseudoprocess (R^n, R, R_*, u) which we shall denote shortly by u .

A pseudoprocess (X, R, R_*, v) , where X is a topological space, is said to be a *limit pseudoprocess* for a pseudoprocess (X, R, R_*, u) , if there exists a sequence $\{T_n\}$, $T_n \in R$, such that $T_n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} u^{t+T_n}(x, \tau) = v^t(x, \tau)$, for every $t \in R$, $x \in X$, $\tau \in R_*$ (see definition of limit process in [1]).

Assume in the sequel that $f: R \times R^n \rightarrow R^n$ is a continuous function such that the differential equation (1) gives a pseudoprocess u .

THEOREM. Suppose that a sequence $\{T_n\}$, $T_n \in R$ has the property

$$(2) \quad f(t + T_n, x) \rightarrow g(t, x)$$

uniformly on compact sets in R and uniformly for $x \in R^n$, where the function g fulfils the Lipschitz condition with respect to x . Then the differential equation

$$(3) \quad x' = g(t, x)$$

defines a pseudoprocess (we denote this pseudoprocess by v), such that for every $x \in R^n$

$$(4) \quad u^{t+T_n}(x, \tau) \rightarrow v^t(x, \tau), \quad \text{as } n \rightarrow \infty$$

uniformly on compact sets in $R \times R_*$.

If the convergence in (2) is uniform in R , then the convergence in (4) is also uniform in R . In the case $T_n \rightarrow \infty$, v is a limit pseudoprocess for u .

Proof. We show first that for every $x \in R^n$ there exists $\lim_{n \rightarrow \infty} u^{t+T_n}(x, \tau)$ and the convergence is uniform on compact sets in $R \times R_*$.

Let $[A_1, A_2]$ and $[0, B_1]$ ($A_1, A_2, B_1 \in R, B_1 > 0$) be any intervals of R and R_* respectively and let $x \in R^n$, $\varepsilon > 0$ be fixed. We may assume that $[A_1, A_2] \cup [0, B_1]$ is contained in an interval $[A, B]$, where $A, B \in R$. Let $K > 0$ be the Lipschitz constant for function g . In the sequel we put

$$p_{n,m}(t, \tau) = |u^{t+T_n}(x, \tau) - u^{t+T_m}(x, \tau)|$$

for every $n, m \in N$ and $t \in R, \tau \in R_*$. We have

$$\begin{aligned} p_{n,m}(t, \tau) &= |\varphi_{(t+T_n, x)}(t+T_n+\tau) - \varphi_{(t+T_m, x)}(t+T_m+\tau)| \\ &= \left| \int_{t+T_n}^{t+T_n+\tau} f(s, \varphi_{(t+T_n, x)}(s)) ds - \int_{t+T_m}^{t+T_m+\tau} f(s, \varphi_{(t+T_m, x)}(s)) ds \right| \\ &\leq \int_t^{t+\tau} |f(s+T_n, \varphi_{(t+T_m, x)}(s+T_n)) - g(s, \varphi_{(t+T_n, x)}(s+T_n))| ds + \\ &\quad + \int_t^{t+\tau} |f(s+T_m, \varphi_{(t+T_m, x)}(s+T_m)) - g(s, \varphi_{(t+T_m, x)}(s+T_m))| ds + \\ &\quad + K \int_0^\tau p_{n,m}(t, s) ds. \end{aligned}$$

It follows from (2) that there exists $n_0 \in N$ such that

$$p_{n,m}(x, \tau) \leq \varepsilon e^{-KB_1} + K \int_0^\tau p_{n,m}(t, s) ds$$

for $n, m \geq n_0$ and $t \in [A_1, A_2], \tau \in [0, B_1]$. By the known result (see p. 67 in [5]) we have

$$p_{n,m}(t, \tau) \leq \varepsilon e^{-KB_1} e^{K\tau} \leq \varepsilon$$

for the same n, m and t, τ . Thus the sequence $\{u^{t+T_n}(x, \tau)\}$ fulfils the Cauchy condition and we have for every $x \in R^n$

$$v^{t+T_n}(x, \tau) \rightarrow v^t(x, \tau)$$

uniformly on the compact sets in $R \times R_*$ for some function $v: R \times R^n \times R_* \rightarrow R^n$. If the convergence in (2) is uniform on R , then we may write " $t \in R$ " in place of " $t \in [A_1, A_2]$ ", so that this convergence is uniform on R .

Now we show that v is the pseudoprocess defined by the differential equation $x' = g(t, x)$. Since

$$\begin{aligned} & |f(t+T_n+\tau, u^{t+T_n}(x, \tau)) - g(t+\tau, v^t(x, \tau))| \\ & \leq |f(t+T_n+\tau, u^{t+T_n}(x, \tau)) - g(t+\tau, u^{t+T_n}(x, \tau))| + \\ & \quad + K|u^{t+T_n}(x, \tau) - v^t(x, \tau)| \end{aligned}$$

we have

$$\frac{d}{d\tau} u^{t+T_n}(x, \tau) \rightarrow g(t+\tau, v^t(x, \tau)), \quad \text{as } n \rightarrow \infty$$

uniformly on the compact sets in R_* . Hence the function $R_* \ni \tau \rightarrow v^t(x, \tau) \in R^n$ is differentiable and its derivative is equal $g(t+\tau, v^t(x, \tau))$. This proves that

$$\psi_{(t_0, x_0)}: [t_0, \infty) \ni s \rightarrow v^t(x, s-t_0) \in R^n$$

is the solution of the problem $x' = g(t, x)$, $x(t_0) = x_0$. By the Lipschitz condition for g it is exactly one solution of this problem. Thus v is the pseudoprocess generated by the equation (3).

The proof of the Theorem is completed.

Let X be a topological space. We say a pseudoprocess (X, R, R_*, u) is *compact* (compare with [1], [2]), if and only if for every sequence $\{T_n\}$, $T_n \in R$ there exists a subsequence $\{T_{a_n}\}$ such that $u^{t+T_{a_n}}(x, \tau)$ is convergent for every $t \in R$, $x \in R^n$, $\tau \in R_*$ (moreover, if this convergence is uniform on R , then u is said to be uniformly compact pseudoprocess).

We obtain from Theorem the following

COROLLARY 1. *Assume that f fulfils the Lipschitz condition with respect to x , and for every $\{T_n\}$, $T_n \in R$ there is $\{T_{a_n}\}$ such that $f(t+T_{a_n}, x)$ is convergent uniformly on the compact sets in R and uniformly on R^n . Then the pseudoprocess u is compact. Moreover, if this convergence is uniform on R , then u is uniformly compact.*

Remark. The last two assumptions coincide with the assumption of the almost periodicity of the function $f(\cdot, x): R \ni t \rightarrow f(t, x) \in R^n$ for an arbitrarily fixed point $x \in R^n$. (see [3], [6]).

From Corollary 1 we have as an immediate consequence

COROLLARY 2. *If f is an almost periodic function and satisfies the Lipschitz condition with respect to x , then u is an uniformly compact process.*

We have also

COROLLARY 3. Assume that the function f is bounded and uniformly continuous. If the pseudoprocess u is uniformly compact, then $f(\cdot, x)$ is an almost periodic function for every $x \in R^n$.

Proof. It is known that the limit of the uniformly convergent sequence of almost periodic functions is an almost periodic function. We fix a point $x \in R^n$ and define the functions:

$$f_n(t) = n[\varphi_{(t,x)}\left(t + \frac{1}{n}\right) - \varphi_{(t,x)}(t)] \quad \text{for } n \in N, t \in R.$$

As the process is uniformly compact then the functions f_n are almost periodic. We have

$$(5) \quad |f_n(t) - f(t, x)| = \left| n \int_t^{t + \frac{1}{n}} f(s, \varphi_{(t,x)}(s)) - f(t, \varphi_{(t,x)}(t)) ds \right| \\ \leq n \int_t^{t + \frac{1}{n}} |f(s, \varphi_{(t,x)}(s)) - f(t, \varphi_{(t,x)}(t))| ds.$$

As the function f is bounded then the function $\varphi_{(t,x)}(\cdot)$ satisfies the Lipschitz condition with a constant not depending on t . Hence, by the uniform continuity of the function f we obtain: for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $t \in R$ and $|s_1 - s_2| < \delta$, we have

$$|f(s_1, \varphi_{(t,x)}(s_1)) - f(s_2, \varphi_{(t,x)}(s_2))| < \varepsilon.$$

We may use this condition in inequality (5) to finish the proof.

References

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