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DCLXI UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXIV 1984

Limits of algebraic sets of bounded degree

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Abstract. The main result of this paper states that the set of all pure dimensional algebraic subsets of C^n of bounded degree is compact in the topology of the local uniform convergence.

1. Introduction. Let Ω be an open subset of C^n. Let $\mathscr{F}_\Omega$ be the family of all closed subsets of Ω. We endow $\mathscr{F}_\Omega$ with the topology $\mathscr{T}_\Omega$ generated by the sets

$$\mathscr{U}(S, K) = \{F \in \mathscr{F}_\Omega\colon\ F \cap K = \emptyset,\ F \cap U \neq \emptyset \text{ for } U \in S\}$$

corresponding to all compact subsets $K \subset \Omega$ and all finite families $\mathscr{S}$ of open subsets of Ω. We call this topology *the topology of local uniform convergence.*

Let us recall some properties of the topology $\mathscr{T}_\Omega$ (for details see [7]).

(1) $\mathscr{F}_\Omega$ is a metrizable compact space;

(2) a closed subset F of Ω is the limit of the sequence $\{F_\nu\}$ in the above topology if and only if the following two conditions hold

i) for every $x \in F$ there exists a sequence $x_\nu \in F_\nu$, $\nu = 1, 2, \ldots$, such that $x_\nu \to x$.

ii) for every compact subset $K \subset \Omega \setminus F$, $F_\nu \cap K = \emptyset$ for sufficiently large ν.

E. Bishop [1] (see also [3] and [6]) has proved the following theorem that will be useful to us.

THEOREM 1 (Bishop). *Let $\{V_\nu\}$ be a sequence of purely k-dimensional analytic subsets of an open subset $\Omega \subset C^n$ which $\mathscr{T}_\Omega$-converges to a (non-empty) limit set V in Ω. If for every compact set $K \subset \Omega$, $2k$-volumes of $V_\nu \cap K$ are uniformly bounded then V is again a purely k-dimensional analytic subset of Ω.*

2. Sequences of algebraic sets. We will denote by $\mathscr{V}_k$ the family of all pure k-dimensional algebraic subsets of C^n, the empty set included, for $k = 1, \ldots, n$. Furthermore, we endow $\mathscr{V}_k$ with the topology induced from $\mathscr{F}_{C^n}$. By $\deg V$ we mean the degree, in the sense of algebraic geometry (see [4], Chapter 5), of the projective completion of $V \in \mathscr{V}_k$, $V \neq \emptyset$ and $\deg \emptyset = 0$ (by definition).

PROPOSITION 1. *The function*

$$\mathscr{V}_{\nu} \ni V \to \deg V \in \boldsymbol{Z}$$

is lower-semicontinuous.

Proof. We see at once that this function is lower-semicontinuous at $\emptyset$. Let $V_\nu \in \mathscr{V}_k$ for $\nu = 0, 1, \ldots$, be non-empty algebraic sets and let $V_\nu \to V_0$. Standard methods of algebraic geometry give the following information. There exists an affine $n-k$ dimensional subspace L of $\boldsymbol{C}^n$ such that

$$\#(L \cap V_\nu) = \deg V_\nu \quad \text{for } \nu = 0, 1, \ldots .$$

It follows from [7], Theorem 3 that $L \cap V_\nu \to L \cap V_0$. Hence $\deg V_\nu \geqslant \deg V_0$ for sufficiently large ν. This concludes the proof of Proposition 1.

Let k, d be two non-negative integers. Write

$$\mathscr{V}_{k,d} = \{V \in \mathscr{V}_k \colon \deg V \leqslant d\}.$$

Now, we state the main result of this paper

THEOREM 2. $\mathscr{V}_{k,d}$ *is a compact subset of* $\mathscr{V}_k$.

Proof. One should prove that the set $\mathscr{V}_{k,d}$ is closed in $\mathscr{F}_{\boldsymbol{C}^n}$. Let $V_\nu \in \mathscr{V}_{k,d}$, $\nu = 1, 2, \ldots$, and let $V_\nu \to V$, where V is a non-empty closed subset of $\boldsymbol{C}^n$. It follows from [2], Theorem 1.8 that

$$\mathrm{Vol}_{2k}(V_\nu \cap B(0, r)) \leqslant \alpha(2k) d \cdot r^{2k} \quad \text{for } r > 0,\ \nu \geqslant 1$$

(where $\alpha(2k)$ denotes $\mathscr{L}^{2k}$-volume of the unit ball in $\boldsymbol{R}^{2k}$). In particular the sets V_ν, $\nu = 1, 2, \ldots$, have uniformly bounded $2k$-volumes on compact subsets of $\boldsymbol{C}^n$. Therefore by Theorem 1 the set V is purely k-dimensional analytic subset of $\boldsymbol{C}^n$. Applying [6], Proposition 3, p. 24 to sequences $V_\nu \cap B(0, r) \to V \cap B(0, r)$, $r > 0$ we get a positive constant K such that

$$\mathrm{Vol}_{2k}(V \cap B(0, r)) \leqslant K \cdot r^{2k} \quad \text{for } r > 0.$$

By known Stoll's criterion ([5], see also [6]) V is algebraic. It is easy to see that the Proposition implies that $\deg V \leqslant d$. Hence $V \in \mathscr{V}_{k,d}$.

Let us end with

COROLLARY (cp. [4], Chapter 2). *Let* $p\colon \boldsymbol{C}^n = \boldsymbol{C}^{n-k} \times \boldsymbol{C}^k \to \boldsymbol{C}^k$ *be the natural projection. Let* V *be a purely dimensional algebraic subset of* $\boldsymbol{C}^n$. *Then*

(1) $\overline{p(V)}$ *is an algebraic subset of* $\boldsymbol{C}^k$ *and* $\dim \overline{p(V)} \leqslant \dim V$. ($\overline{p(V)}$ *is the closure of* $p(V)$ *in the topology of* $\boldsymbol{C}^k$).

(2) *if the restriction* $p|V\colon V \to \boldsymbol{C}^k$ *is proper then* $p(V)$ *is a purely dimensional algebraic subset of* $\boldsymbol{C}^k$, $\dim p(V) = \dim V$ *and* $\deg p(V) \leqslant \deg V$.

Proof. For each $\lambda \in \mathbf{C}\setminus\{0\}$ let

$$p_\lambda\colon \mathbf{C}^n = \mathbf{C}^{n-k} \times \mathbf{C}^k \ni (x, y) \to (\lambda x, y) \in \mathbf{C}^n$$

be a scalar multiplication of x by λ. Since the mappings p_λ are linear isomorphisms, the sets $p_\lambda(V)$ are algebraic with $\deg p_\lambda(V) = \deg V$, for $\lambda \neq 0$. This implies that $p_\lambda(V) \in \mathscr{V}_{s,d}$ where $s = \dim V$, $d = \deg V$, for $\lambda \neq 0$. By Theorem 2 there exists a sequence $\lambda_\nu \to 0$, $\lambda_\nu \neq 0$ such that the sequence $p_{\lambda_\nu}(V)$ converges in $\mathscr{F}_{\mathbf{C}^n}$ to a non-empty limit set W which is a purely s-dimensional algebraic subset of $\mathbf{C}^n$ with $\deg W \leqslant d$. Note also that $W \cap (\{0\} \times \mathbf{C}^k) = \{0\} \times \overline{p(V)}$. Furthermore, if $p|V$ is proper then $W = \{0\} \times p(V)$. This concludes the proof of Corollary.

References

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Received January 10, 1981.