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Discontinuous Solutions of an Inhomogeneous Linear Functional Equation

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1. We consider the iterative functional equation

$$\varphi(f(x)) = g(x)\varphi(x)+h(x) \tag{1}$$

in an interval $I := (0, A)$ or $(0, A]$, where $0<A\leqslant+\infty$. We denote

$$I^* := I \cup \{0\}.$$

We deal with the same problem as in the paper [3] where we have studied the homogeneous equation

$$\varphi(f(x)) = g(x)\varphi(x), \tag{2}$$

looking for asymptotic properties of these of its solutions which are discontinuous at the origin. In [3] we aimed at finding a characterization of all solutions of (2) continuous in I. In fact, the results of the papers [3] and [1] almost settle the problem of asymptotic behaviour (as $x \to 0+$) of such solutions of (2).

Because of the form of the general solution of the inhomogeneous equation (1) we should then look for a particular solution of (1) with known asymptotic properties. If (1) has a solution which is continuous in I^*, then the question is settled by the results of [1] and [5]. Thus in the present paper we are mostly interested in the case, where equation (1) does not possess any continuous solution in I^*.

In the sequel all the asymptotic symbols in this paper refer to $x \to 0+$.

2. We accept the following hypotheses on the given functions f, g, and h:

(H) f, g, h: $I^* \to \mathbf{R}$ are continuous in I^* and $0<f(x)<x$, $g(x)>0$ in I. Moreover, there are positive constants $a, b, c, m, k, q, \mu, \varkappa$ such that the following relations hold as $x \to 0+$:

$$f(x) = x-ax^{m+1}+0(x^{m+1+\mu}),$$

$$g(x) = 1+bx^k+0(x^{k+\varkappa}),$$

$$h(x) = cx^q+o(x^q).$$

Hypotheses (H) imply (cf. [3], [2], and also [4], ch. II) that for equation (2) two cases are possible, if we look for solutions continuous in I^*: if $k>m$, then (A) there is a one-parameter family of such solutions, while if $k\leqslant m$, then (C) the function $\varphi(x)=0$ for $x\in I^*$ is the only continuous solution of (2) in I^*. The case (B) $C(I^*)$ — solution of (2) depends on an arbitrary function corresponds to the condition $b<0$ and $k\leqslant m$ in (H) and will be treated elsewhere.

3. We shall use the following lemma which we quote after [5] in the form adapted to equation (1).

LEMMA 1. *Let hypotheses* (H) *be fulfilled. If $k\leqslant m\leqslant q$ but $k<q$, then equation* (1) *has the only continuous solution $\varphi_0\colon I^*\to \boldsymbol{R}$ and this function has the asymptotic property*

(3) $$\varphi_0(x) = 0(x^{q-k}),$$

and, if $k<m$, even better one

(4) $$\varphi_0(x) = (c/b)x^{q-k}+o(x^{q-k}).$$

Now we are going to prove the following theorem (note that we impose no condition in q if $k<m$).

THEOREM 1. *Let hypotheses* (H) *be fulfilled. If $k\leqslant m$ then equation* (1) *has the only solution $\varphi_0\colon I\to \boldsymbol{R}$, continuous in I and possessing the property* (4) *if $k<m$, and the property* (3) *if $k=m$ and $q>m-b/a$.*

Proof. We shall look for continuous in I^* solutions of the auxiliary equation

(5) $$\psi(f(x)) = g_1(x)\psi(x)+h_1(x),$$

where

$$g_1(x) := \big(f(x)/x\big)^p g(x) \quad \text{for } x\in I,\ g_1(0) := a^p,$$
$$h_1(x) := \big(f(x)\big)^p h(x) \quad \text{for } x\in I^*$$

and p is a number to be determined later. A solution ψ of (5) is connected with the φ of (1) through the formula

(6) $$\psi(x) = x^p\varphi(x) \quad \text{for } x\in I.$$

By (H) we have

$$g_1(x) = (1-ax^m+0(x^{m+\mu}))^p(1+bx^k+0(x^{k+\varkappa})) = 1+bx^k-pax^m+0(x^\delta),$$

where $\delta=\min(2m, m+\mu, k+\varkappa)>0$, i.e. (since $k\leqslant m$):

(7) $$g_1(x) = 1+b_1x^k+0(x^{k+\varkappa_1})$$

with some positive $\varkappa_1$ and

(8) $$b_1 := \begin{cases} b & \text{if } k < m, \\ b-pa & \text{if } k = m. \end{cases}$$

Moreover,

$$h_1(x) = cx^{p+q}(1-ax^m+0(x^{m+\mu}))^p(1+o(1)) = cx^{p+q}+o(x^{p+q}).$$

Thus we see that the given functions f, g_1 and h_1 in equation (5) fulfil all the hypotheses (H).

Now, let be $k<m$. Then we have $b_1 = b$ (cf. (8)) and $\delta>k$, so that

$$\varkappa_1 = \min(\delta-k, m-k)>0.$$

Next, we can always take a p to fulfil $p+q\geqslant m$. Thus lemma 1 applies to equation (5) yielding the existence of the only continuous solution $\psi_0\colon I^* \to \boldsymbol{R}$ which has the property

$$\psi_0(x) = c/bx^{p+q-k}+o(x^{p+q-k}).$$

This means that the function φ_0 connected with the ψ_0 by formula (6) is the only solution of equation (1) having the properties stated in the theorem (in particular, the asymptotic property (4)).

In the case $k = m$ we proceed similarly as before. Now we have in formula (7) (cf. (8)) $b_1 = b-pa$, $\varkappa_1 = \min(m, \mu, \varkappa)>0$. Since we assumed $m-q<b/a$, one can take a p to fulfil $m-q<p<b/a$. For such a p we have $b_1>0$ and $p+q>m = k$ and lemma 1 applies to equation (5). We get the only continuous in I^* solution Ψ_0 of equation (5) possessing the property (cf. (3) and $k = m$)

$$\psi_0(x) = 0(x^{p+q-m})$$

which shows in turn (similarly as in the preceding case) the existence of the only solution $\varphi_0\colon I \to \boldsymbol{R}$ of (1) which enjoys the property (3). This completes the proof of the theorem.

4. Combining results from the paper [3] on asymptotic properties of solutions of equation (2) with that of the preceding section we get the following

THEOREM 2. *Let hypotheses* (H) *be fulfilled. If* $k<m$ *and* $\min(k, \varkappa, \mu)>m-k$ *then every continuous in* I *solution* φ *of equation* (1) *has the asymptotic property*

(9) $$\varphi(x) = c/bx^{q-k}+o(x^{q-k})+0(\exp(b/a(m-k)x^{k-m}).$$

If $k = m$ *and* $q>m-b/a$ *then every solution* φ *of equation* (1) *which is continuous in* I *has the asymptotic property*

(10) $$\varphi(x) = 0(x^{q-m}).$$

Proof. Every solution φ of (1) is of the form $\varphi = \varphi_0+\Phi$ where φ_0 is the particular solution of (1) determined by theorem 1 and Φ is a solution of (2). Considering the case $k<m$ we see that the asymptotic formula (9) results from (4) and the property of every

solution of (2) continuous in I (cf. [3], th. 3). If $k = m$ in this way we obtain from (3) and theorem 2 in [3] the formula

$$\varphi(x) = 0(x^{q-m}) + 0(x^{-b/a}).$$

But $q-m > -b/a$, so $0(x^{q-m}) \subset 0(x^{-b/a})$ and property (10) follows.

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