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On Order and Type of Meromorphic Functions

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1. Introduction. Let f be a function holomorphic in a neighbourhood of infinity except the point $z = \infty$. We define the *order* of f at infinity as follows:

$$\varrho(f) = \varrho = \inf\{\mu>0\colon\ \exists R_0 \forall R>R_0 M_f(R)<\exp(R^\mu)\},$$

where $M_f(R) = \|f\|_{C(0,R)} = \sup\{|f(z)|\colon\ |z| = R\}$. If $\varrho(f)$ is positive but smaller than infinity, we define the *type* of f at infinity as follows:

$$\sigma(f) = \sigma = \inf\{\nu>0\colon\ \exists R_0 \forall R>R_0 M_f(R)<\exp(\nu R^\varrho)\}.$$

We can equivalently define:

$$\varrho(f) = \limsup_{R\to\infty} \frac{\ln^+\ln^+ M_f(R)}{\ln R}$$

and

$$\sigma(f) = \limsup_{R\to\infty} \frac{\ln M_f(R)}{R^\varrho}, \quad \text{if} \quad 0<\varrho<\infty.$$

The following properties of ϱ and σ can be proved:

1° $\varrho(f+g)\leqslant\max(\varrho(f), \varrho(g))$ for any functions f and g holomorphic in a neighbourhood of infinity;

2° if $0<\varrho(f) = \varrho(g) = \varrho(f+g)<\infty$, then $\sigma(f+g)\leqslant\max(\sigma(f), \sigma(g))$;

3° if p is a nonzero polynomial, then:

$$\varrho(p) = 0 \tag{1.1}$$

$$\varrho(f+p) = \varrho(f\cdot p) = \varrho(f) \tag{1.2}$$

$$\sigma(f+p) = \sigma(f\cdot p) = \sigma(f) \quad \text{if } 0<\varrho(f)<\infty. \tag{1.3}$$

For details see e.g. [5].

In this paper we will restrict our interest to the class of meromorphic functions. Denote by $M_m(\boldsymbol{C})$ the class of all meromorphic functions with exactly m poles (counted with their multiplicities) on the complex plane $\boldsymbol{C}$. The purpose of this paper is to generalize the results obtained by Winiarski [9] for the class $M_0(\boldsymbol{C})$ of all entire functions onto the class $M_m(\boldsymbol{C})$, $m\geqslant 0$.

2. An extension of the Batyrev-Winiarski theorem. Let $\mathcal{R}_{n,m}$ denote the class of all rational functions whose numerators and denominators are polynomials of degree not greater than n and m, respectively. For every function f defined on the compact set E there exists a function $f_n \in \mathcal{R}_{n,m}$ such that

$$\|f-f_n\|_E = \inf\{\|f-r\|_E:\ r \in \mathcal{R}_{n,m}\},$$

where $\|\varphi\|_E = \sup\{|\varphi(z)|:\ z \in E\}$. We define:

$$\varrho_{n,m}(f, E) = \varrho_{n,m} = \|f-f_n\|_E,$$

$$E_n(f, E) = \varrho_{n,0}(f, E) = \inf\{\|f-p\|_E:\ p \in \mathcal{P}_n\},$$

where $\mathcal{P}_n$ denotes the space of all polynomials of degree not greater than n. Winiarski [9] has proved the following theorem: *Let E be a compact subset of* **C** *with the logarithmic capacity $d>0$. Let f be an entire function. Then*:

1° $\limsup\limits_{n\to\infty} n^{\frac{1}{\varrho}}(E_n(f, E))^{\frac{1}{n}} = d(e\varrho\sigma)^{\frac{1}{\varrho}}$ *if* $0<\varrho(f)<\infty$,

2° $\lim\limits_{n\to\infty} n^{\alpha}(E_n(f, E))^{\frac{1}{n}} = 0$ *for every* $\alpha>0$ *if* $\varrho(f) = 0$.

We shall give the following generalization of this theorem:

THEOREM 2.1. *Let $E\subset$* **C** *be a compact set with the logarithmic capacity $d>0$. Let f be a function of the class $M_m(\mathbf{C})$. If* $0<\varrho(f)<\infty$, then

(2.1) $$\limsup_{n\to\infty} n^{\frac{1}{\varrho}}(\varrho_{n,m})^{\frac{1}{n}} = d(e\varrho\sigma)^{\frac{1}{\varrho}}.$$

If $\varrho(f) = 0$, then

(2.2) $$\lim_{n\to\infty} n^{\alpha}(\varrho_{n,m})^{\frac{1}{n}} = 0 \quad \textit{for every } \alpha>0.$$

Proof. Since f belongs to the class $M_m(\mathbf{C})$, there exists an entire function g and a polynomial q such that

(2.3) $$f(z) = \frac{g(z)}{q(z)}$$
$$q(z) = (z-\zeta_1)\cdot\ldots\cdot(z-\zeta_m).$$

The points $\zeta_1, \ldots, \zeta_m$ are not necessarily distinct. If for any j the point ζ_j lies in E, then the principal parts of the expansions of f and of the approximants f_n at ζ_j are all equal. That means that

(2.4) $$\varrho_{n,m}(f, E) = \varrho_{n-s,m-s}(\tilde{f}, E),$$

where s is the multiplicity of the pole of f at ζ_j and $\tilde{f}$ is the difference of f and the principal part of the Laurent expansion of f at ζ_j. It follows from (1.2) and (1.3) that the order and the type of f are equal to those of f. This fact, together with (2.4), allows us to assume that f is holomorphic on E.

The following inequality holds:

$$\varrho_{n,m} \leqslant \frac{E_n(g, E)}{\min\limits_{z \in E} |q(z)|}. \tag{2.5}$$

Hence

$$\limsup_{n \to \infty} n^{\alpha}(\varrho_{n,m})^{\frac{1}{n}} \leqslant \limsup_{n \to \infty} n^{\alpha}(E_n(g, E))^{\frac{1}{n}} \quad \text{for } \alpha > 0. \tag{2.6}$$

The equality (2.2) follows immediately from (2.6) and from the Winiarski theorem. If $0 < \varrho(f) < \infty$, then, as a consequence of (2.6) and the Winiarski theorem, we get

$$\limsup_{n \to \infty} n^{\frac{1}{\varrho}}(\varrho_{n,m})^{\frac{1}{n}} \leqslant d(e\varrho\sigma)^{\frac{1}{\varrho}}.$$

Assume now that the opposite inequality is not true. Then we can choose a number $\sigma' \in (0, \sigma)$ such that

$$\varrho_{n,m} < d^n \left(\frac{e\varrho\sigma'}{n}\right)^{\frac{n}{\varrho}} \quad \text{for almost every } n.$$

Then

$$\|f_{n+1} - f_n\|_E < 2d^n \left(\frac{e\varrho\sigma'}{n}\right)^{\frac{n}{\varrho}}, \quad n \geqslant N_0. \tag{2.7}$$

Let $f_n(z) = \dfrac{p_n(z)}{q_n(z)}$, where p_n and q_n are polynomials of degree not greater than n and m, respectively, and q_n is of the form

$$q_n(z) = q_{n,0} + q_{n,1}z + \ldots + q_{n,m_n-1}z^{m_n-1} + z^{m_n}, \quad m_n \leqslant m.$$

Walsh [8] has proved that $m_n = m$ when n is large enough and that q_n converges to q uniformly on every compact set.

Let $\delta > 0$ be smaller than the distance from ζ_j to E for each j. Set

$$D = \{z \in \mathbf{C}\colon |z - \zeta_j| \geqslant \delta,\ 1 \leqslant j \leqslant m\}.$$

There exists an integer N_1 and a number $c > 0$ such that

$$|q_n(z)| \geqslant c^{-1} \quad \text{if } z \in D \text{ and } n \geqslant N_1.$$

Hence we derive, for $z \in D$ and $n \geqslant N_1$,

$$|f_n(z) - f_{n-1}(z)| \leqslant c^2 |p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z)|. \tag{2.8}$$

The convergence of the sequence q_n implies that

$$\|q_n\|_E \leqslant M \quad \text{for every } n, \tag{2.9}$$

where M is a positive constant. Then we derive from (2.7) and (2.9) that

$$\|p_n q_{n-1} - p_{n-1} q_n\|_E \leqslant 2M^2 d^n \left(\frac{e\varrho\sigma'}{n}\right)^{\frac{n}{\varrho}} \quad \text{for } n > N_0.$$

Let $\sigma'' \in (\sigma', \sigma)$. Then there exists an integer N_2 such that

(2.10) $$\|p_n q_{n-1} - p_{n-1} q_n\|_E \leqslant d^{n+m} \left(\frac{e\varrho\sigma''}{n+m}\right)^{\frac{n+m}{\varrho}} \quad \text{for } n \geqslant N_2 .$$

It follows from (2.8) that

(2.11) $$|f(z)| \leqslant |f_{N_1}(z)| + c^2 \sum_{n=N_1+1}^{\infty} |p_n(z) q_{n-1}(z) - p_{n-1}(z) q_n(z)|$$

for $z \in D$. Repeating the proof of lemma 3.3. (cf. [9]) we derive from (2.10) and (2.11) that either the order of f is smaller than ϱ or the type of f is smaller than σ, what contradicts the assumptions of the theorem.

PROPOSITION 2.1. *Under the assumptions of the previous theorem, the order of f satisfies the equality*

(2.12) $$\varrho(f) = \limsup_{n\to\infty} \frac{-n \ln n}{\ln \varrho_{n,m}} .$$

Proof. Put $\mu_0 = \limsup\limits_{n\to\infty} \dfrac{-n\ln n}{\ln \varrho_{n,m}}$. Assume that $\mu_0 < \infty$. Fix μ and μ' such that $\mu_0 < \mu' < \mu$. Then $(\varrho_{n,m})^{\frac{1}{n}} < \left(\frac{1}{n}\right)^{\frac{1}{\mu'}} d \left(\frac{1}{n}\right)^{\frac{1}{\mu}}$ for almost every n.

Put $\sigma = (e\mu)^{-1}$. Then

(2.13) $$(\varrho_{n,m})^{\frac{1}{n}} < d \left(\frac{e\mu\sigma}{n}\right)^{\frac{1}{\mu}} \quad \text{for almost every } n.$$

Using the same argument as in the proof of theorem 2.1 we get $\varrho(f) \leqslant \mu$ as a consequence of (2.13). But μ has been chosen arbitrarily, so $\varrho(f) \leqslant \mu_0$. This inequality remains valid if μ_0 is infinite.

On the other side, (1.2) implies that

(2.14) $$\varrho(f) = \varrho(g) ,$$

where g is defined by (2.3). Hence

(2.15) $$\varrho(f) = \limsup_{n\to\infty} \frac{\ln n}{\ln \dfrac{d}{(E_n(g, E))^{\frac{1}{n}}}}$$

(cf. [9], theorem (4.2). Comparing (2.15) with (2.5) we get the inequality

$$\limsup \frac{-n\ln n}{\ln \varrho_{n,m}(f, E)} \leqslant \varrho(f)$$

and, consequently, $\mu_0 = \varrho(f)$, what ends the proof.

PROPOSITION 2.2. *Let* $0<\mu<\varrho(f)$. *Then*

$$\limsup_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m})^{\frac{1}{n}} = +\infty .$$

Proof. Suppose that $n^{\frac{1}{\mu}}(\varrho_{n,m})^{\frac{1}{n}}<A<\infty$, $n = 0, 1, 2, \ldots$ Then

$$\varrho_{n,m}<\left(\frac{A^{\mu}}{n}\right)^{\frac{n}{\mu}} = d^n\left(\frac{e\nu\mu}{n}\right)^{\frac{n}{\mu}},$$

where ν is a positive constant. Hence $\varrho(f)\leqslant\mu$, as we have shown in the proof of proposition 2.1. This contradicts the assumptions.

PROPOSITION 2.3. *Let the function* f *be of finite order. Let* $\mu>\varrho(f)$. *Then* $\lim\limits_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m})^{\frac{1}{n}} = 0.$

Proof. Let $\mu' \in (\varrho(f), \mu)$. Then

$$\limsup_{n\to\infty} n^{\frac{1}{\mu'}}(E_n(g, E))^{\frac{1}{n}}\leqslant d(e\mu')^{\frac{1}{\mu'}}$$

(see [9], the proof of theorem 4.1.) Owing to this fact and to the inequality (2.6), $n^{\frac{1}{\mu'}}(\varrho_{n,m}(f, E))^{\frac{1}{n}}<A$, where A does not depend on n. Consequently,

$$\lim_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m})^{\frac{1}{n}}\leqslant A \lim_{n\to\infty} n^{\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)} = 0 ,$$

what ends the proof.

Before stating the converse of theorem 2.1, we shall prove the following

LEMMA 2.1. *Let* $f\in M_{m'}(C)$ *satisfy the condition*

$$\limsup_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m}(f, E))^{\frac{1}{n}}\leqslant d(e\mu\nu)^{\frac{1}{\mu}} , \tag{2.16}$$

where μ, ν *are positive numbers and* $m\geqslant m'$. *Then for every* $K>\nu$, $M_f(R)\leqslant\exp(KR^{\mu})$, *when* R *is large enough.*

Proof. Let Φ be the extremal function of the set E (see [4]). Put

$$E_R = \{z\in C\colon \quad \Phi(z)\leqslant R\},\ \Gamma_R = \{z\in C\colon \quad \Phi(z) = R\} \text{ for } R>1.$$

Denote $\hat{M}_f(R) = \|f\|_{\Gamma_R}$. It is sufficient to prove that if $K>\nu$, then $\hat{M}_f(R) \leqslant\exp(K(dR)^{\mu})$, when R is large enough (cf. [9], lemma 3.1.) Without loss of generality we may assume that the function f has no poles in the set E.

Choose a number K', $\nu<K'<K$. By (2.16), there exists an integer n_0 such that

(2.17) $$\varrho_{n,m}\leqslant d^{n+4m}\left(\frac{eK'\mu}{n+4m}\right)^{\frac{n+4m}{\mu}} \quad \text{for } n\geqslant n_0 .$$

Put $k = m-m'$. It follows from [3] that the best approximant f_n is of the form

$$f_n(z) = \frac{p_n(z)}{q_{1,n}(z)q_{2,n}(z)} = \frac{p_n(z)}{q_n(z)},$$

where $p_n\in\mathscr{P}_n$, $q_{1,n}\in\mathscr{P}_{m'}$, $q_{2,n}\in\mathscr{P}_k$ and the zeros of $q_{1,n}$ tend to the poles of f when $n\to\infty$.

Assume that $q_{i,n}(z) = \prod_j (z-z_j^{(i,n)})$ for $i = 1, 2$. Then $\lim_{n\to\infty} q_{1,n}(z) = q(z)$, where the polynomial q is defined as in (2.3), with m replaced by m'.

Choose $R_0>1$ such that for every n, $|q_{1,n}(z)|\geqslant 1$ if $\Phi(z)\geqslant R_0$. There exists a constant $M>0$ such that $\|q_{1,n}\|_E\leqslant M$ for $n\in N$.

Fix $R\geqslant R_0$. Let z belong to E_R. Then

$$|f_{n+1}(z)-f_n(z)| = \frac{|p_{n+1}(z)q_n(z)-p_n(z)q_{n+1}(z)|}{|q_n(z)q_{n+1}(z)|}$$
$$\leqslant \frac{\|q_n\|_E\cdot\|q_{n+1}\|_E}{|q_n(z)|\cdot|q_{n+1}(z)|}\cdot\|f_{n+1}-f_n\|_E\cdot R^{n+m+1} .$$

Hence

(2.18) $$|f_{n+1}(z)-f_n(z)|\leqslant 2M^2\varrho_{n,m}\cdot R^{n+m+1}\cdot\frac{\|q_{2,n}\|_E\cdot\|q_{2,n+1}\|_E}{|q_{2,n}(z)|\cdot|q_{2,n+1}(z)|} .$$

Choose $R_1\geqslant R_0$ such that Γ_R is a Jordan curve for $R\geqslant R_1$. Let $z_1^{(2,n)}, z_2^{(2,n)}, \ldots, z_{k_n}^{(2,n}$ be the zeros of $q_{2,n}$. Fix a number $\theta\in(0, 1)$. Put

$$Q_\theta = \bigcup_{n=n_0}^{\infty}\bigcup_{j=1}^{k_n} B(z_j^{(2,n)}, \theta^n),$$

where $B(a, r) = \{z\colon |z-a|<r\}$. Put $d_\theta = \sum_{n=n_0}^{\infty} 2k_n\cdot\theta^n$. Then Q_θ is covered by a family of circles whose sum of diameters is equal d_θ. Hence for every $R\geqslant R_1$ there exists a Jordan curve $\gamma_{R,\theta}$, containing E_R in its interior, such that the distance between each point of $\gamma_{R,\theta}$ and E_R is not greater than d_θ and $\gamma_{R,\theta}$ does not intersect the set Q_θ.

For each R we denote $R_\theta = \inf\{r\colon \gamma_{R,\theta}\subset E_r\}$. Then

$$\lim_{R\to\infty}\frac{R}{R_\theta} = 1 .$$

This equality is a simple consequence of the well-known fact that

$$\lim_{z\to\infty}\frac{\Phi(z)}{|z|} = \frac{1}{d}$$

(cf. [4]). Indeed, we have for $z_R \in \Gamma_R$ and $z_{R_\theta} \in \Gamma_{R_\theta}$

$$\frac{R}{R_\theta} = \frac{\dfrac{\Phi(z_R)}{|z_R|}}{\dfrac{\Phi(z_{R_\theta})}{|z_{R_\theta}|}} \cdot \frac{|z_R|}{|z_{R_\theta}|} .$$

The points z_R and z_{R_θ} can be chosen in such a way that

$$|z_R - z_{R_\theta}| \leqslant d_\theta .$$

Then

$$\lim_{R\to\infty} \frac{R}{R_\theta} = \lim \frac{|z_R|}{|z_{R_\theta}|} = 1 .$$

Denote

$$\operatorname{diam} E = \sup\{|x-y|:\ x, y \in E\} ,$$

$$\operatorname{dist}(z, E) = \inf\{|z-x|:\ x \in E\} .$$

Let $z \in \Gamma_R$. Then (cf. [9], lemma 2.1):

$$\operatorname{dist}(z, E) \leqslant dR \leqslant \operatorname{dist}(z, E) + \operatorname{diam} E .$$

Let $R \geqslant R_2 := \max\left(R_1, \dfrac{\operatorname{diam} E}{d}\right)$. Let $z \in \gamma_{R,\theta}$, $x \in E$. If $z_j^{(2,n)} \in E_{4R_\theta}$, then

(2.19) $$\frac{|x - z_j^{(2,n)}|}{|z - z_j^{(2,n)}|} \leqslant \frac{\operatorname{diam} E + 4dR_\theta}{\theta^n} \leqslant \frac{5dR_\theta}{\theta^n} .$$

If $z_j^{(2,n)} \in C \setminus E_{4R_\theta}$, then

(2.20) $$\frac{|x - z_j^{(2,n)}|}{|z - z_j^{(2,n)}|} \leqslant \frac{d \cdot \Phi(z_j^{(2,n)}) + \operatorname{diam} E}{d \cdot \Phi(z_j^{(2,n)}) - \operatorname{diam} E - (dR + \operatorname{diam} E)} \leqslant \frac{2d \cdot \Phi(z_j^{(2,n)})}{d \cdot \Phi(z_j^{(2,n)}) - 3dR_\theta} \leqslant 8 .$$

Let $R \geqslant R_3 := \max\left(\dfrac{1}{d}, R_2\right)$. Let $z \in \gamma_{R,\theta}$. By (2.18), (2.19) and (2.20) we have

(2.21) $$\|f_{n+1}(z) - f_n(z)\| \leqslant A_1 \varrho_{n,m} \theta^{-nk} (R_\theta)^{n+4m},$$

where A_1 does not depend on n, R and θ.

Choose $R_4 \geqslant R_3$ such that $|q_{n_0}(z)| \geqslant 1$ if $\Phi(z) \geqslant R_4$. Then

(2.22) $$|f_{n_0}(z)| \leqslant |p_{n_0}(z)| \leqslant \|p_{n_0}\|_E \cdot R^{n_0} = A_2 R^{n_0}$$

if $z \in \Gamma_R$, $R \geqslant R_4$.

Fix $R \geqslant R_4$. It has been shown in [3] that $f(z) = \lim\limits_{n\to\infty} f_n(z)$, if z is not a limit point of poles of the functions f_n. Hence, if $z \in \gamma_{R,\theta}$, then

(2.23) $$|f(z)| \leqslant |f_{n_0}(z)| + \sum_{n=n_0}^{\infty} |f_{n+1}(z) - f_n(z)| .$$

Put $n_R = A_3(R_0)^\mu$, where $A_3 = 1+[eK'\mu(2d\theta^{-k})^\mu]$ and $[a] = \max(Z \cap (-\infty, a])$. There exists $R_5 \geqslant R_4$, which depends only on θ, such that $n_R \geqslant n_0$ for $R \geqslant R_5$. Then it follows from (2.17) and (2.21) that $\sum_{n=n_R}^{\infty} |f_{n+1}(z)-f_n(z)| \leqslant A_1$, when $z \in \gamma_{R,\theta}$.

We derive from (2.17) that for arbitrary n

$$\varrho_{n,m}(\theta^{-k}\cdot R_\theta)^{n+4m} \leqslant \exp\left(K'(\theta^{-k}dR_\theta)^\mu\right).$$

This, together with (2.21), (2.22) and (2.23), implies that

$$\|f\|\gamma_{R,\theta} \leqslant A_2(R_\theta)^{n_0} + A_1\left(1+A_3(R_\theta)^\mu\right)\exp\left(K'(\theta^{-k}dR_\theta)^\mu\right).$$

Hence, we can choose θ so close to 1 and a number R_6, such that

$$\hat{M}_f(R) \leqslant \|f\|\gamma_{R,\theta} \leqslant \exp\left(K(dR)^\mu\right) \quad \text{it } R \geqslant R_6,$$

what ends the proof.

THEOREM 2.2. *Let f be a function defined on the set E. Put*

$$S = \{\mu>0\colon \limsup_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m}(f, E))^{\frac{1}{\mu}} < \infty\}.$$

If the set S is not empty, then there exists a number $m' \leqslant m$, such that $\hat{f}$ can be extended to a function $f \in M_{m'}(C)$. Moreover, the function f is of order $\tilde{\varrho} = \inf S$ and if $\tilde{\varrho}>0$, so f is of type $\tilde{\sigma}$, where $\limsup_{n\to\infty} n^{\frac{1}{\tilde{\varrho}}}(\varrho_{n,m})^{\frac{1}{n}} = d(e\tilde{\varrho}\tilde{\sigma})^{\frac{1}{\tilde{\varrho}}}$.

Proof. The existence of f is a consequence of Saff's theorem ([3], [7]). It follows from propositions 2.2 and 2.3 that the order of f is given by the equality:

$$\varrho(f) = \inf\{\mu>0\colon \limsup_{n\to\infty} n^{\frac{1}{\mu}}(\varrho_{n,m'})^{\frac{1}{n}} < \infty\}.$$

Since $\varrho_{n,m} \leqslant \varrho_{n,|m'}$, we get that $\tilde{\varrho} \leqslant \varrho(f)$. On the other side, lemma 2.1 implies that $\mu \geqslant \tilde{\varrho} \Rightarrow \mu \geqslant \varrho(f)$. Hence $\varrho(f) = \tilde{\varrho}$.

Suppose now that $\varrho(f)>0$. Then, by theorem 2.1 and the inequality: $\varrho_{n,|m} \leqslant \varrho_{n,m'}$, we get that $\sigma(f) \geqslant \tilde{\sigma}$. The opposite inequality follows from lemma 2.1.

3. The Newton–Padé approximants. Let $(z_k)_{k=1}^{\infty}$ be a sequence of points on the complex plane. Denote

$$\omega_0(z) = 1,$$

$$\omega_{n+1}(z) = \omega_n(z)\cdot(z-z_{n+1}), \quad z \in \boldsymbol{C}.$$

Let f be a function holomorphic in a neighbourhood of the set

$$\{z_k\colon k = 1, 2, \ldots\}.$$

The (n, m)-th *Newton–Padé approximant* of the function f with respect to the sequence (z_k) is a function $f_{n,m} \in \mathscr{R}_{n,m}$ such that the difference $f - f_{n,m}$ is given by a formal power series:

$$f(z) - f_{n,m}(z) = \sum_{k=n+m+1}^{\infty} d_k \omega_k(z) . \tag{3.1}$$

The function $f_{n,m}$ may not exist but if it exists, then it is unique. In the special case when $z_k = 0$ for each k, the functions $f_{n,m}$ are called the *Padé approximants* of the function f. For details see [1] and [2].

LEMMA 3.1. *Let f be a function of the class $M_m(\boldsymbol{C})$, whose poles are $\zeta_1, \ldots, \zeta_m$, not necessarily distinct. Let $(z_k)_{k=1}^{\infty}$ be a bounded sequence of points in $\boldsymbol{C} \setminus \{\zeta_1, \ldots, \zeta_m\}$. Then, for almost every n, there exists the Newton–Padé approximant $f_{n,m}$ of the function f with respect to the sequence (z_k). The poles of $f_{n,m}$ tend to the respective poles of f, if n tends to infinity. Moreover, $\lim_{n\to\infty} f_{n,m}(z) = f(z)$, uniformly on every compact subset of $\boldsymbol{C} \setminus \{\zeta_1, \ldots, \zeta_m\}$.*

This lemma is only a slight modification of the Saff theorem [6], so we omit the proof.

THEOREM 3.1. *Let $f \in M_m(\boldsymbol{C})$. With the notations as above, let the functions $f_{n,m}$ be of the form:*

$$f_{n,m}(z) = \frac{P_n(z)}{Q_n(z)},$$

where $P_n(z) = \sum_{i=0}^{n} p_{ni} z^i$ and $Q_n(z) = (z - \zeta_{n,1}) \cdot \ldots \cdot (z - \zeta_{n,m_n})$, $m_n \leqslant m$, $n \geqslant n_0$. Then

$$\varrho(f) = \limsup_{n\to\infty} \frac{-n \ln n}{\ln |p_{nn}|} . \tag{3.2}$$

Proof. Lemma 3.1 implies that $\lim_{n\to\infty} Q_n(z) = q(z)$, where q is defined by (2.3). Hence there exists R_0 such that

$$|Q_n(z)| \geqslant 1 \quad \text{for } |z| \geqslant R_0 . \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$|f_n(z) - f_{n-1}(z)| = \left| \frac{p_{nn} \omega_{n+m}(z)}{Q_{n-1}(z) Q_n(z)} \right| \leqslant |p_{nn} \omega_{n+m}(z)| . \tag{3.4}$$

For every $\varepsilon > 0$ there exists $R_\varepsilon \geqslant R_0$ such that

$$\|\omega_{n+m}\|_{C(0,R)} \leqslant ((1+\varepsilon)R)^{n+m} \quad \text{for } R \geqslant R_\varepsilon . \tag{3.5}$$

We derive from (3.4) and (3.5) that

$$M_f(R) \leqslant \|f_{n_0, m}\|_{C(0,R)} + \sum_{n=n_0+1}^{\infty} |p_{nn}| ((1+\varepsilon)R)^{n+m}, \tag{3.6}$$

if $R \geqslant R_\varepsilon$.

Hence, by a standard argument we get the inequality:

$$\varrho(f) \leqslant \limsup_{n\to\infty} \frac{-n\ln n}{\ln|p_{nn}|}.$$

Suppose now that $\varrho(f)<\mu$. Let R be greater than $s := \sup\{z_k: k = 1, 2, 3, \ldots\}$. It follows from (3.1) and the Hermite formula that

$$p_{nn} = \frac{1}{2\pi i} \int_{C(0,R)} \frac{g(z)\,Q_n(z)}{\omega_{n+m+1}(z)}\,dz. \tag{3.7}$$

Let $|z| = R$. Then $|\omega_{n+m+1}(z)| \geqslant (R-s)^{n+m+1}$. By lemma 3.1. there exists a constant C such that for almost all n

$$\|Q_n\|_{C(0,r)} \leqslant Cr^m \quad \text{if } r\geqslant 1.$$

Hence we derive by (3.7) that

$$|p_{nn}| \leqslant C\cdot\exp(R^{\mu})\,R^{m+1}(R-s)^{-n+m+1}.$$

Put $R = \left(\frac{n}{\mu}\right)^{\frac{1}{\mu}}$. Then it follows directly that

$$\limsup_{n\to\infty} \frac{-n\ln n}{\ln|p_{nn}|} \leqslant \mu.$$

Hence

$$\limsup \frac{-n\ln n}{\ln|p_{nn}|} \leqslant \varrho(f),$$

what ends the proof of (3.2).

In similar way as the previous theorem we can prove the following

THEOREM 3.2. 1° *If* $0<\varrho(f)<\infty$ *and* f *is of type* σ, *then*

$$\limsup_{n\to\infty} n^{\frac{1}{\varrho}}|p_{nn}|^{\frac{1}{n}} = (e\varrho\sigma)^{\frac{1}{\varrho}};$$

2° *If* $\varrho(f) = 0$, *then for every* $\alpha>0$,

$$\lim_{n\to\infty} n^{\alpha}|p_{nn}|^{\frac{1}{n}} = 0.$$

Suppose now that for every k, $z_k = 0$. Denote:

$$C(k,l) = \begin{vmatrix} a_k & a_{k-1} & \cdots & a_{k-l+1} \\ a_{k+1} & a_k & \cdots & a_{k-l+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k+l-1} & a_{k+l-2} & \cdots & a_k \end{vmatrix},$$

where $f(z) = \sum_{i=0}^{\infty} a_i z^i$. There are known some algorithms which enable us to derive the determinants $C(k, l)$ in an easy way (see [1]). Due to the well-known equality:

$$|p_{mn}| = \left|\frac{C(n, m+1)}{C(n+1, m)}\right|,$$

we can restate the last two theorems in a somewhat different form:

PROPOSITION 3.1. *Let $f \in M_m(C)$ be holomorphic at zero. Then (provided $C(n, m)$ are different from zero for almost every n):*

1° $\varrho(f) = \limsup_{n\to\infty} \frac{-n\ln n}{\ln|C(n, m+1)| - \ln|C(n+1, m)|}$;

2° *If $\varrho(f) = 0$, then for every $\alpha > 0$:*

$$\limsup_{n\to\infty} n^\alpha \left|\frac{C(n, m+1)}{C(n+1, m)}\right|^{\frac{1}{n}} = 0;$$

3° *If $0 < \varrho(f) < \infty$ and the type of f is equal to σ, then*

$$\limsup_{n\to\infty} n^{\frac{1}{\varrho}} \left|\frac{C(n, m+1)}{C(n+1, m)}\right|^{\frac{1}{n}} = (e\varrho\sigma)^{\frac{1}{\varrho}}.$$

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