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On Mosco-type Characterization of (K)-convergence and Lower Semicontinuity of Integral Functionals

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1. Introduction. The so-called (K)-property of Kuratowski and (Q)-property of Cesari have been intensively investigated in optimal control theory in connection with the lower closure property of orientor fields or lower semicontinuity (l.s.c.) of integral functionals (see [3], [11] and the references in there). It appears (see [1], [2], [5], [6], [9]: compare also Remark 3.2 in this paper) that these notions are also closely related to the Γ- and G-convergences defined by E. De Giorgi ([4]) and S. Spagnolo ([14]).

In the present paper after preliminaries of Section 2 we characterize, in Section 3, the (K)-semicontinuity of a net of sets (i.e. the (K)-property of a net, see [6]) by the conditions of Mosco type ([10]). This result is used, in Section 4, to obtain a characterization of the l.s.c. of the integral functional

$$\mathscr{F}(z, u) = \int_T h(t, z, u)dt\,.$$

Finally, in Section 5, we give a theorem (Th. 5.1) providing some sufficient conditions for $\mathscr{F}$ to be l.s.c. This theorem when specified (see Th. 5.2 and Th. 5.3) contains some results of C. Olech (see Th. 3 and Th. 3′ in [11]) obtained in the other way. Namely, in papers [3], [11] the orientor fields $Q(t, z) = \operatorname{epi} h(t, z, \cdot)$ were considered and some good behaviour of the set of selectors of $Q(t, z(t))$ (when $z(\cdot)$ changes) was needed for l.s.c. of $\mathscr{F}$. In this paper we consider rather functions $f_z(\cdot) = \mathscr{F}(z, \cdot)$ and l.s.c. of $\mathscr{F}$ is characterized by the (K) property of the nets $\{\operatorname{epi} f_{z_\iota}\}_J$ (see Th. 4.1 and Corollary 4.1). In the proof of Th. 5.1 we make use of the Rockfellar characterization of integral functionals by their conjugates ([12]) and of some characterizations of the property (Q) (which is stronger than the property (K)) given in the author's paper [6].

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2. Preliminaries. The notation and the notions used throughout this paper are explained in detail in paper [6]. We recall some of them for the convenience of the reader. Let $I \neq \emptyset$ denote a directed set. Without any loss of generality we may assume that I

possesses the greatest element ω; that is $\omega \succ \iota$ for every $\iota \in I$. We shall write in the sequel $\bar{I} = I \cup \{\omega\}$, $\omega \notin I$, and the letter J will be used for both cases $J = \bar{I}$ or $J = I$.

By a net $m = \{M_\iota\}_J$ in a set $X(\{M_\iota\}_J \subset X)$ we mean a function defined on a directed set J with values in X; i.e. $m\colon J \to X$. Given a net $\pi = \{\Pi_\nu\}_N$ in a set Y and a directed subset J of Y we say π is cofinal in J iff for every $\iota_0 \in J$ there is $\nu_0 \in N$ such that

$$\pi(\{\nu \in N | \nu \succ \nu_0) \subset \{\iota \in J | \iota \succ \iota_0\} .$$

A subset J' of J is called cofinal in J iff the net $\pi = i = \{\mathrm{Id}_\iota\}_{J'}$ $(N = J'$, $\mathrm{Id} =$ identity$)$ is cofinal in J; this amounts to saying that for every $\iota \in J$ there is $\iota' \in J'$ such that $\iota' \succ \iota$. In the case $J = \bar{I}$ we agree, in addition, to call cofinal in J each subset J' which is cofinal in I, so in consequence a cofinal subset of $\bar{I}$ need not contain ω. We denote by $\mathscr{J}$ the family of all cofinal subsets of J. A cofinal subset J' of J is called residual in J iff $J'' = J \setminus J'$ is not cofinal in J; this amounts to saying that there is $\iota' \in J$ such that $J' \supset \{\iota \in J | \iota \succ \iota'\}$. Similarly, a net $\pi = \{\Pi_\nu\}_N$ cofinal in a directed set J is called residual in J iff for every $\nu_0 \in N$ the set $J \cap \{\Pi_\nu | \nu > \nu_0\}$ is residual in J.

Now we are in a position to formulate

DEFINITION 2.1. Given two nets $m = \{m_\iota\}_J$ and $\pi = \{\Pi_\nu\}_N$ such that $\pi(N) \subset J$ we say that the net $m \circ \pi = \{M_{\Pi_\nu}\}_N$ is a *cofinal* (residual) *subnet* of m iff the net π is cofinal (residual) in J.

DEFINITION 2.2. We say that a net $\{x_\iota\}_J$ in a topological space (X, τ) is *convergent* to an element $x \in X$ and we write $x = \tau\text{-}\lim\limits_J x_\iota$ iff for every neighbourhood U of x the set $J'_U = \{\iota \in J | x_\iota \in U\}$ is residual in J (this means that for every U there is ι'_U such that $x_\iota \in U$ for $\iota \succ \iota_U$).

Let 2^X denote the family of all subsets of a topological space (X, τ). Given a net $\{M_\iota\}_J$ in 2^X and a set $U \subset X$ we put $J_U = \{\iota \in J | U \cap M_\iota \neq \emptyset\}$ and we adopt the following definition (compare [8], [7], [6]).

DEFINITION 2.3. The sets

$$(K)\text{-}\limsup_J M_\iota = \{x \in X |\ \text{for every neighbourhood } U \text{ of } x \text{ the set } J_U \text{ is cofinal in } J\}$$

$$(K)\text{-}\liminf_J M_\iota = \{x \in X |\ \text{for every } U \text{ as above } J_U \text{ is residual in } J\}$$

are called, respectively, the *upper* and the *lower limits* in Kuratowski sense (or (K)-limits) of the net $\{M_\iota\}_J$. If both (K)-limits coincide we say the net $\{M_\iota\}_J$ is (K)-*convergent*: i.e.

$$M = (K)\text{-}\lim_J M_\iota \ \text{ iff } \ (K)\text{-}\liminf_J M_\iota = (K)\text{-}\limsup_J M_\iota = M .$$

If in X several topologies are given we shall indicate the considered topology τ writing $(K)_\tau$-instead of (K)-. Frolik ([7]) has given the following two characterizations of the above limits:

PROPOSITION 2.1. *Under the above notation* $x \in (K)\text{-}\limsup_J M_\iota$ $((K)\text{-}\liminf_J M_\iota)$ *iff there exist a cofinal* (residual) *subnet* $\{M_{\Pi_\nu}\}_N$ *of the net* $\{M_\iota\}_J$ *and points* $x_\nu \in M_{\Pi_\nu}$ *such that* $x = \tau\text{-}\lim_N x_\nu$.

PROPOSITION 2.2. *Denoting by* $\mathrm{cl}\,A$ *the closure of a set* $A \subset X$ *we have*

$$(K)\text{-}\limsup_J M_\iota = \bigcap_{\lambda \neq \omega} \mathrm{cl} \bigcup_{\iota > \lambda} M_\iota$$

$$(K)\text{-}\liminf_J M_\iota = \bigcap_{J' \in \mathcal{J}} (K)\text{-}\limsup_{J'} M_\iota$$

Hence we get

COROLLARY 2.1. *The upper and lower* (K)*-limits of the net* $\{M_\iota\}_J \subset 2^X$ *as well as the* (K)*-limit* (*if it exists*) *are closed subsets of* X.

Assume now X is totally ordered and that the order is linear (i.e. $x \geqslant y$, $y \geqslant x$ implies $x = y$). Then for a net $\{m_\iota\}_J$ in X and the topology τ given by the order we adopt

DEFINITION 2.4. If the following elements

$$\limsup_J m_\iota := \sup\{(K)\text{-}\limsup_J \{m_\iota\}\}$$

$$\liminf_J m_\iota := \inf\{(K)\text{-}\limsup_J \{m_\iota\}\}$$

exist, then they are called, respectively, the *upper* and the *lower limit* of the net $\{m_\iota\}_J$.

Remark 2.1. The upper and lower limits of the net $\{m_\iota\}_J$ exist provided the order in X is total linear and continuous (i.e. each bounded from above subset of X has supremum in X); for instance it is the case $X = \boldsymbol{R}$ or $X = \overline{\boldsymbol{R}}$ ($\boldsymbol{R}$ denotes the reals and $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$).

From Proposition 2.1 and Definition 2.4 we get

COROLLARY 2.2. *If the upper and lower limits of the net* $\{m_\iota\}_J$ *exist, they are given by the formulae*:

$$\limsup_J m_\iota = \sup\{\lim_N m_{\Pi_\nu} \mid \{m_{\Pi_\nu}\}_N \textit{ is a convergent cofinal subnet of the net } \{m_\iota\}_J\}$$

$$\liminf_J m_\iota = \inf\{\textit{the set as above}\}.$$

Moreover, they are always attained, since the set $(K)\text{-}\limsup_J \{m_\iota\}$ *is closed.*

Finally, we have

Remark 2.2. The net $\{m_\iota\}_J$ in X is convergent (in the sense of Definition 2.2) iff $\liminf_J m_\iota = \limsup_J m_\iota$.

3. The Mosco type characterization of (K)-convergence. Throughout this section we assume (X, τ) is a topological space, f and f_ι $(\iota \in J)$ are functions defined on X with values in $\bar{\boldsymbol{R}}$. By the epigraph of a function f as above we mean the set

$$K_f = \operatorname{epi} f = \{(x, \beta) \in X \times \boldsymbol{R} | \beta \geqslant f(x)\}.$$

The Carthesian product $X \times \boldsymbol{R}$ is supplied with the product topology $\mathscr{T} = \tau \times \tau_{\boldsymbol{R}}$ ($\tau_{\boldsymbol{R}}$ denotes the natural topology in $\boldsymbol{R}$). Under the notation $K_\iota = K_{f_\iota}$ we have

LEMMA 3.1. *The conditions* (1) *and* (i) *below are equivalent*

(1) $$(K)_{\mathscr{T}}\text{-}\limsup_J K_\iota \subset K_f$$

(i) *For every cofinal subnet* $\{K_{\Pi_\nu}\}_N$ *of the net* $\{K_\iota\}_J$ *and the points* $x_\nu \in X$ *such that* $x = \tau\text{-}\lim\limits_N x_\nu$ *we have*

$$\liminf_N f_{\Pi_\nu}(x_\nu) \geqslant f(x).$$

Whereas the following condition

(2) $$(K)_{\mathscr{T}}\text{-}\liminf_J K_\iota \supset K_f$$

follows from

(ii) *For every* $x \in X$ *there exist a residual subnet* $\{K_{\Pi_\nu}\}_N$ *of the net* $\{K_\iota\}_J$ *and the points* $x_\nu \in X$ *such that* $x = \tau - \lim\limits_N x_\nu$ *and*

$$\limsup_N f_{\Pi_\nu}(x_\nu) \leqslant f(x).$$

Moreover, the conditions (2) *and* (ii) *are equivalent provided either* $f(x) > -\infty$ *for every* $x \in X$ *or the topology* τ *satisfies the first axiom of denumerability.*

From this lemma, owing to the fact that residual subnet is also cofinal, we get immediately.

COROLLARY 3.1. *If either* $f(x) > -\infty$ *for every* $x \in X$ *or the topology* τ *satisfies the first axiom of denumerability, then* $K_f = (K)\text{-}\lim K_{f_\iota}$ *iff the condition*

(i) *of the Lemma* 3.1 *and the following condition*

(ii′) *For every* $x \in X$ *there exist a residual subnet* $\{K_{f_{\Pi_\nu}}\}_N$ *of the net* $\{K_{f_\iota}\}_J$ *and the points* $x_\nu \in X$ *such that* $x = \tau\text{-}\lim\limits_N x_\nu$ *and*

$$f(x) = \lim_N f_{\Pi_\nu}(x_\nu)$$

are satisfied.

Remark 3.1. The characterization given in Corollary 3.1 was obtained by Mosco ([10]) in the case of a denumerable sequence of convex and lower semi-continuous functions defined on a Banach space.

Proof of Lemma 3.1. The part (1) $\Leftrightarrow$ (i) and (ii) $\Rightarrow$ (2) of the lemma can be proved in a similar way as in Mosco paper [10]. So we confine ourselves only to the most difficult part of the proof; i.e. the implication (2) $\Rightarrow$ (ii).

Let $x \in X$. The case $f(x) = +\infty$ being trivial, assume $f(x)$ is finite. Then $(x, f(x)) \in K_f$ and by the assumption (2) and Proposition 2.1 we find a residual subnet $\{K_{\Pi_\nu}\}_N$ of $\{K_\iota\}_J$ and points $(x_\nu, \beta_\nu) \subset K_{\Pi_\nu}$ (so $\beta_\nu \geqslant f_{\Pi_\nu}(x_\nu)$) such that

$$(x, f(x)) = \mathscr{T}\text{-}\lim_N (x_\nu, \beta_\nu).$$

This means that $x = \tau\text{-}\lim_N x_\nu$ and $f(x) = \tau_{\boldsymbol{R}}\text{-}\lim_N \beta_\nu = \limsup_N \beta_\nu \geqslant \limsup_N f_{\Pi_\nu}(x_\nu)$, so the required condition (ii) is satisfied which proves the implication in the case $f(x) > -\infty$ for every $x \in X$.

Let us notice that in the remaining case when $f(x) = -\infty$ for a point $x \in X$ the whole "vertical" line $\{x\} \times \boldsymbol{R}$ lies in the set K_f. So by condition (2) $\{x\} \times \boldsymbol{R} \subset (K)\text{-}\liminf K_\iota$. Hence, in virtue of Proposition 2.1 for the point $(x, -n)$, where n is an integer ($n \in N$), there exist a residual subnet $\{K_{\Pi^n_\nu}\}_{N_n}$ of $\{K_\iota\}_J$ and points $(x^n_\nu, \beta^n_\nu) \in K_{\Pi^n_\nu}$ (so $\beta^n_\nu \geqslant f_{\Pi^n_\nu}(x^n_\nu)$) $\nu \in N_n$, such that

$$(x, -n) = \mathscr{T}\text{-}\lim_{N_n} (x^n_\nu, \beta^n_\nu). \tag{3.1}$$

By the assumed first axiom of denumerability we have a decreasing sequence $\{U_n\}_N$ of neighbourhoods of the point x at our disposal (i.e. $U_n \supset U_{n+1}$ $n = 1, 2, \ldots$). Therefore, from (3.1) it follows that there is $\nu^n_0 \in N_n$ such that

$$x^n_\nu \in U_n, \quad \beta^n_\nu \leqslant -n + \frac{1}{n} \quad \text{for } \nu \in N_n,\ \nu \succ \nu^n_0. \tag{3.2}$$

Owing to the fact that the nets $\{\Pi^n_\nu\}_{\nu \in N_n}$ $n = 1, 2, \ldots$ are residual in J, the ν^n_0 $n = 1, 2, \ldots$ can be chosen in such a way that

$$\{\Pi^n_\nu \mid \nu \in N_n,\ \nu \succ \nu^n_0\} \supset \{\iota \in J \mid \iota \succ \Pi^n_{\nu^n_0} \tag{3.3}$$

$$\Pi^{n+1}_\nu \succ \Pi^n_{\nu^n_0} \quad \text{for } \nu \in N_{n+1},\ \nu \succ \nu^{n+1}_0; \qquad \Pi^{n+1}_{\nu^{n+1}_0} \neq \Pi^n_{\nu^n_0}. \tag{3.4}$$

Now, let us observe that the set $N = \bigcup_{n \in \boldsymbol{N}} N^0_n$, where $N^0_n = \{\nu \in N_n \mid \nu \succ \nu^n_0\}$, is directed by the following relation: $\nu_2 \succ \nu_1$ iff $\nu_1, \nu_2 \in N^0_n$ and $\nu_2 \succ \nu_1$ in the directed set N_n or $\nu_2 \in N^0_{n_2}$, $\nu_1 \in N^0_{n_1}$ and $n_2 > n_1$. Clearly $\{K_{\Pi_\nu}\}_N$, where $\Pi_\nu = \Pi^n_\nu$ for $\nu \in N^0_n$, is a residual subnet of $\{K_\iota\}_J$ (for $\{\Pi_\nu \mid \nu \in N,\ \nu \succ \nu^n_0\} \supset \{\iota \in J \mid \iota \succ \Pi^n_{\nu^n_0}\}$). Thus, setting $x_\nu = x^n_\nu$, $\beta_\nu = \beta^n_\nu$ if $\Pi^n_{\nu^n_0} \prec \prec \Pi^n_\nu \prec \Pi^{n+1}_{\nu^{n+1}_0}$, we get from (3.2)

$$x = \tau\text{-}\lim_N x_\nu, \qquad \limsup_N f_{\Pi_\nu}(x_\nu) \leqslant \limsup_N \beta_\nu \leqslant -n + \frac{1}{n} \quad \text{for } n \in \boldsymbol{N}.$$

This means that $\limsup_N f_{\Pi_\nu}(x_\nu) = -\infty$ which completes the proof of condition (ii).

We end this section with the following

Remark 3.2. It is worth noticing that the conditions (i) and (ii′) of Corollary 3.1 specified for a denumerable sequence of functions defined on a topological space satisfying the first axiom of denumerability characterize the so-called Γ^--convergence (see 3.1 Proposizione in [15] and Proposition 1.2 in [9]). So in this case the Γ^--convergence: $f = \Gamma^-(\tau)\text{-}\lim_N f_n$ (i.e.

$$f(x) = \sup_{U\in\mathscr{U}_x} \liminf_N \inf_{x\in U} f_n(x) = \sup_{U\in\mathscr{U}_x} \limsup_N \inf_{x\in U} f_n(x) \quad \text{for } x\in X,$$

where $\mathscr{U}_x$ denotes a basis of neighbourhoods of x) means simply the (K)-convergence of epigraphs; i.e.

$$\operatorname{epi} f = (K)_{\mathscr{T}}\text{-}\lim_N \operatorname{epi} f_n \quad (\mathscr{T} = \tau\times\tau_R).$$

4. Lower semicontinuity of integral functionals. Throughout this section we admit the following.

Assumption H_1: A function $h\colon T\times \boldsymbol{R}^n\times \boldsymbol{R}^m \to \boldsymbol{R}\cup\{+\infty\}$ is $\mathscr{L}\times\mathscr{B}$ measurable, where $\mathscr{L}$ and $\mathscr{B}$ denote, respectively, the σ-algebras of Lebesgue measurable subsets of the interval $T = [0, 1]$ and of the Borel subsets of $\boldsymbol{R}^{n+m}$.

Then for any pair of measurable functions $z\colon T\to \boldsymbol{R}^n$ and $u\colon T\to \boldsymbol{R}^m$ the function $T\ni t \to h(t, z(t), u(t)) \in \boldsymbol{R}\cup\{+\infty\}$ is measurable and we set

$$\mathscr{F}(z, u) = \begin{cases} \int_T h(t, z(t), u(t))dt & \text{if the integrand is an inegrable function} \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

So $\mathscr{F}(z, u)$ belongs to $\overline{\boldsymbol{R}}$ for any pair (z, u) as above.

We shall consider the functions z and u belonging to the fixed sets $z\in\mathscr{C}$ and $u\in F$. Here $\mathscr{C} = \mathscr{C}(T)$ denotes a class of measurable functions with a fixed topology τ; usually $\mathscr{C}$ will be the class $C(T, \boldsymbol{R}^n)$ of all $\boldsymbol{R}^n$-valued continuous functions on T with the topology of the uniform or the pointwise convergence. Similarly, F denotes a space of measurable functions on T with values in $\boldsymbol{R}^m$ such that there exists another space G of measurable functions which together with F form a dual system. For instance

$$F = L^p = L^p(T, \boldsymbol{R}^m) \;\; 1\leqslant p<\infty, \;\; G = L^q, \;\; \text{if } p>1, \; \frac{1}{p}+\frac{1}{q} = 1,$$

and

$$G = S\subset L^\infty(T, \boldsymbol{R}^m) \quad \text{if } p = 1 \;(S \text{ denotes } L^\infty \text{ or } C \text{ or } C^\infty).$$

On the space F we consider the weak $\sigma = \sigma(F, G)$ topology.

DEFINITION 4.1. We say the functional $\mathscr{F}$ is l.s.c. at the point $(z_\omega, u_\omega) \in \mathscr{C} \times F$ with respect to the product topology $\tau \times \sigma$ iff for every net $\{(z_\iota, u_\iota)\}_J \subset \mathscr{C} \times F$ such that

$$z_\omega = \tau\text{-}\lim_J z_\iota \tag{4.2}$$

$$u_\omega = \sigma\text{-}\lim_J u_\iota \tag{4.3}$$

we have

$$\mathscr{F}(z_\omega, u_\omega) \leqslant \liminf_J \mathscr{F}(z_\iota, u_\iota)\,. \tag{4.4}$$

We say $\mathscr{F}$ is l.s.c. (in $\mathscr{C} \times F$) iff it is l.s.c. at every point $(z_\omega, u_\omega) \in \mathscr{C} \times F$.

Now, by means of the functional $\mathscr{F}$ we define the following family of functions:

$$f_z\colon F \ni u \mapsto f_z(u) := \mathscr{F}(z, u) \in \bar{R} \qquad (z \in \mathscr{C})\,. \tag{4.5}$$

Setting

$$K_z = \operatorname{epi} f_z \tag{4.6}$$

we are in a position to formulate

THEOREM 4.1. *The functional* $\mathscr{F}$ *given by* (4.1) *is* l.s.c. *at* $(z_\omega, u_\omega) \in \mathscr{C} \times F$ *with respect to the topology* $\tau \times \sigma$ *iff for every net* $\{z_\iota\}_J$ *satisfying* (4.2) *we have*

$$(K)_{\mathscr{T}}\text{-}\lim_J \sup K_{z_\iota} \subset K_{z_\omega} \qquad (\mathscr{T} = \sigma \times \tau_R)\,. \tag{4.7}$$

The proof of this theorem is an immediate consequence of the Lemma 3.1, since the Definition 4.1 formulated in terms of the functions f_{z_ι} is clearly equivalent to the condition (i) of the lemma.

In the sequel we admit the function h satisfies also

Assumption H_2: $h(t, z, \cdot)$ is convex for every $(t, z) \in T \times R^n$. This assumption is quite natural in lower semicontinuity theorems, since such a convexity is (in a certain sense) a necessary condition for $\mathscr{F}$ given by (4.1) to be l.s.c. (compare [3], [11]). The Assumption H_2 implies all the sets K_z $(z \in \mathscr{C})$ given by (4.6) are convex subsets of $F \times R$.

Setting $J = \bar{I} = I \cup \{\omega\}$, we adopt (compare [6])

DEFINITION 4.2. We say the net $\{K_{z_\iota}\}_{\bar{I}}$ *has the property* (K) (is (K)-upper semi-continuous) iff the condition (4.7) with $J = \bar{I}$ $(J = I)$ is fulfilled.

Since, by Proposition 4.5 (e) in the author's paper [6] a net $\{K_\iota\}_{\bar{I}}$ of closed and convex sets has the property (K) iff it is (K)-upper semi-continuous, we get from Theorem 4.1 the following

COROLLARY 4.1. *If the function h satisfies the assumptions* H_1 *and* H_2, *then* $\mathscr{F}$ *given by* (4.1) *is* l.s.c. *at* $(z_\omega, u_\omega) \in \mathscr{C} \times F$ *with respect to the* $\tau \times \sigma$ *topology iff for every net* $\{z_\iota\}_I$

such that (4.2) *is fulfilled the net* $\{K_{z_i}\}_{\bar{I}}$ *has the property* (*K*). *This means that in* (4.7) *with* $J = \bar{I}$ *we have actually equality.*

Hence, we immediately obtain

COROLLARY 4.2. *Under the assumptions of Corollary* 4.1, *if* $\mathscr{F}$ *is* l.s.c. *in* $\mathscr{C} \times F$ *with respect to* $\tau \times \sigma$ *topology, then all the sets* K_z $(z \in \mathscr{C})$ *are convex and* $\sigma \times \tau_R$*-closed subsets of* $F \times \boldsymbol{R}$. *This amounts to saying that all the functions* $f_z(\cdot)$ $(z \in \mathscr{C})$ *are convex and* σ- l.s.c. *on F. If, in addition, we assume that for a* $z_0 \in \mathscr{C}$ *there exists* $u_0 \in F$ *such that* $\mathscr{F}(z_0, u_0)$ *is finite, then* $f_{z_0}(u) > -\infty$ *for every* $u \in F$.

The last assertion of the above corollary is due to the fact that a convex and l.s.c. function might be equal to $-\infty$ on a closed and convex set but then it is equal to $+\infty$ elsewhere.

COROLLARY 4.3. *If* $\mathscr{F}$ *does not depend on z, that is if we define*

$$\tilde{\mathscr{F}}(u) = \int_T \tilde{h}(t, u)\,dt$$

where $\tilde{h}(t, \cdot)$ *is convex and there is* $u_0 \in F$ *such that* $\tilde{\mathscr{F}}(u_0)$ *is finite, then* $\tilde{\mathscr{F}}$ *is* σ*-l.s.c. in F iff the set* $K = \operatorname{epi}\tilde{\mathscr{F}}$ *is convex,* $\sigma \times \tau_R$*-closed and bounded from below that is it does not contain a vertical line (i.e. the set* $\{u\} \times \boldsymbol{R}$ *with some* $u \in F$*).*

Remark 4.1. Corollary 4.3 may be compared with a result of C. Olech (see Theorem 1 in [11]) who in this case gave full characterization of the lower semicontinuity.

5. A sufficient condition for l.s.c. of $\mathscr{F}$**.** By $\Gamma(F)$ we denote the space of all regular functions on F with values in $\bar{\boldsymbol{R}}$ (a regular function is the pointwise supremum of a family of affine functions) and we put

$$\Gamma_0(F) = \{f \in \Gamma(F) | f \not\equiv +\infty, f \not\equiv -\infty\}.$$

Thus, a regular function is convex, σ-l.s.c. ($\sigma = \sigma(F, G)$, F and G are as in Section 4) and it does not assume the value $-\infty$ without being constant ($\equiv -\infty$). Similarly, we denote by $\tilde{\Gamma}(F)$ the space of all σ-l.s.c. functions on F with values in $\bar{\boldsymbol{R}}$. For any function f: $F \to \boldsymbol{R}$ we denote by $\tilde{\Gamma}[f]$ and $\Gamma[f]$, respectively, the l.s.c. regularization and the convex and l.s.c. regularization of f. So $\tilde{\Gamma}[f] \in \tilde{\Gamma}(F)$ and $\Gamma[f] \in \Gamma(F)$ for any f as above. By $\mathscr{K}(F)$, $\mathscr{K}_0(F)$ and $\tilde{\mathscr{K}}(F)$ we denote, respectively, the sets of all the epigraphs of functions belonging to $\Gamma(F)$, $\Gamma_0(F)$ and $\tilde{\Gamma}(F)$. Of course, all these definitions may be repeated for the space G. Thus, given a function f; $F \to \boldsymbol{R}$ we find that its conjugate f^* belongs to $\Gamma(G)$, where $f^*(v) = \sup_{u \in F}[\langle u, v\rangle - f(u)]$, $v \in G$, $\langle\cdot, \cdot\rangle$ being the pairing between F and G.

From Section 7 of paper [6] (see formulae (7.4), (7.12) and (7.4′), (7.14)) we get