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Note on Similarity to Contractions

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Abstract. A generalization of Rota's Theorem [4] to the case of countably many operators is presented here. Consequently, we get a „universal model" for such families of operators. From the main result we also obtain a dilation theorem, similar to that of Brehmer [5, Prop. 9.2., p. 39].

The present paper deals with the following problems:

(1) When a commutative sequence of operators in a Hilbert space is similar to a common part of a commutative sequence of backward shifts.

(2) When a commutative sequence of operators in a Hilbert space is similar to a commutative sequence of contractions, which has a unitary dilation.

The problem (1) was first solved for a single operator by G. C. Rota [4]. He showed that every operator with a spectral radius strictly less than 1 is similar to part of a backward shift. In other words, the backward shift may be treated as a "universal model" for such operators. Later D. N. Clark [2] and J. A. Ball [1] obtained similar results for a finite sequence of commuting operators. The present note solves the problems (1) and (2) for a special class of infinite sequences.

1. The construction of the "universal model". In what follows X always stands for the sets $\{1, 2, \ldots, n\}$ or $\{1, 2, \ldots\}$, $(n = 1, 2, \ldots)$. Let Z be the set of all integers. Denote by $F(X)$ the abelian group of all functions $\alpha\colon X \to Z$ with finite supports and pointwise defined addition $+$. $F(X)$ with pointwise defined partial order relation $\geqslant$ becomes a a partially ordered group, i.e. if $\alpha \geqslant \beta$ then $\alpha+\gamma \geqslant \beta+\gamma$, for every $\gamma \in F(X)$. 0 stands for the neutral element of $F(X)$. Let $F_+(X)$ be the set of all elements α of $F(X)$ such that $\alpha \geqslant 0$. Denote by e_m $(m \in X)$ the element of the set $F_+(X)$ defined by

$$e_m(m') = \begin{cases} 1 & \text{for } m' = m \\ 0 & \text{for } m' \neq m \,. \end{cases}$$

Let H be a complex Hilbert space with the inner product $(.,.)$ and the norm $\|h\| = (h, h)^{1/2}$ $(h \in H)$. $L(H)$ stands for the algebra of all bounded linear operators in H.

I_H stands for the identity operator in H. Recall [4] that an operator V in H is a *pure isometry* (or a *unilateral shift*) if and only if V is an isometry satisfying the condition

$$\bigcap_{k=0}^{\infty} V^k H = \{0\}\,.$$

The dimension of the orthogonal complement $H \ominus VH$ of VH in H is called the *multiplicity* of V. A *backward shift* is the adjoint of a unilateral shift.

For a non-empty set A, denote by $l_H^2(A)$ the Hilbert space of all systems $h = (h_\alpha)_{\alpha\in A}$ of vectors $h_\alpha \in H$ such that $\Sigma_\alpha \|h_\alpha\|^2 < +\infty$, with coordinatewise defined linear operations and the inner product defined by the formula

$$(h, k) = \sum_\alpha (h_\alpha, k_\alpha) \quad \text{for } h = (h_\alpha),\ k = (k_\alpha) \in l_H^2(A)\,.$$

The space $l_H^2(F_+(X))$ will be shortly denoted by H_X. We now define the operator V_m in H_X $(m \in X)$ by

(3) $$(V_m h)_\alpha = \begin{cases} h_{\alpha-e_m} & \text{for } \alpha \geqslant e_m \\ 0 & \text{otherwise} \end{cases} \qquad (h \in H_X)\,.$$

PROPOSITION 1. *The operator V_m is a pure isometry. The multiplicity of V_m equals the dimension of $H_m = \{h \in H_X\colon h_\alpha = 0$ for $\alpha \geqslant e_m\}$, and it does not depend on $m \in X$. The adjoint V_m^* of V_m has the form*

$$(V_m^* h)_\alpha = h_{\alpha+e_m} \quad \textit{for } h \in H_X\,.$$

The operators V_m $(m \in X)$ commute with each other.

Proof. V_m is an isometry, because

$$\|V_m h\|^2 = \sum_\alpha \|(V_m h)_\alpha\|^2 = \sum_{\alpha \geqslant e_m} \|h_{\alpha-e_m}\|^2 = \|h\|^2 \quad \text{for } h \in H_X\,.$$

It is plain that

(4) $$V_m^k H_X = \{h \in H_X\colon h_\alpha = 0 \text{ if } \alpha \ngeqslant k e_m\}\,.$$

Hence

(5) $$\bigcap_{k=0}^{m} V_m^k H_X = \{0\}\,.$$

Indeed, if $h \in \bigcap_{k=0}^{\infty} V_m^k H_X$ and $\alpha \in F_+(X)$, then $h \in V_m^{(\alpha(m)+1)} H_X$. Since $\alpha \ngeqslant (\alpha(m)+1)e_m$ we have $h_\alpha = 0$ for all $\alpha \in F_+(X)$. This proves (5), so V_m is a pure isometry. The multiplicity of V_m equals $\dim H_m$, because by (4)

(6) $$H_m = H_X \ominus V_m H_X\,.$$

Now we shall show that $\dim H_m$ does not depend on $m \in X$. To prove this, we define a mapping $\varphi_{m,m'}(m, m' \in X)$ from $F(X)$ onto $F(X)$ by

$$(\varphi_{m,m'}(\alpha))(k) = \begin{cases} \alpha(k) & \text{for } k \in X \setminus \{m, m'\}; \\ \alpha(m') & \text{for } k = m; \\ \alpha(m) & \text{for } k = m'. \end{cases}$$

The mapping $\varphi_{m,m'}$ is a group automorphism which preserves a partial order, that is

$$\alpha \geqslant \beta \quad \text{if and only if } \varphi_{m,m'}(\alpha) \geqslant \varphi_{m,m'}(\beta) \quad \text{for all } \alpha, \beta \in F(X).$$

Moreover $\varphi_{m,m'}(F_+(X)) = F_+(X)$ and $\varphi_{m,m'}(e_m) = e_{m'}$, $\varphi_{m,m'}(e_{m'}) = e_m$.
It is clear that the operator $U_{m,m'}$ in H_X defined by

$$(U_{m,m'}h)_\alpha = h_{\varphi_{m,m'}(\alpha)} \quad \text{for } h \in H_X,$$

is unitary and satisfies the following conditions

$$(7) \qquad U_{m,m'}V_m = V_{m'}U_{m,m'}, \qquad U_{m,m'}V_{m'} = V_m U_{m,m'}.$$

It follows from (6) and (7) that $U_{m,m'}H_m = H_{m'}$, thus $\dim H_m = \dim H_{m'}$.

To find the form of the operator V_m^*, take $h, h' \in H_X$. By a simple computation we get

$$(V_m^* h, h') = (h, V_m h') = \sum_{\alpha \geqslant e_m} (h_\alpha, h'_{\alpha - e_m}) = \sum_\alpha (h_{\alpha + e_m}, h'_\alpha) = (k, h'),$$

where $k = (k_\alpha)$ is defined by $k_\alpha = h_{\alpha + e_m}$. This shows that

$$(8) \qquad (V_m^* h)_\alpha = h_{\alpha + e_m} \quad \text{for all } \alpha \in F_+(X).$$

Moreover, (8) implies that the operators $(V_m)_{m \in X}$ commute with each other. This completes the proof.

2. Properties of the representation $\langle \cdot, T \rangle$. Let H be a complex Hilbert space. Suppose that $T = (T_m)_{m \in X}$ is a commutative sequence of operators in H (i.e. $T_m T_{m'} = T_{m'} T_m$ for each $m, m' \in X$). For $\alpha \in F_+(X)$ we put

$$\langle \alpha, T \rangle = \begin{cases} \prod_{m:\alpha(m) \neq 0} T_m^{\alpha(m)} & \text{for } \alpha \neq 0; \\ I_H & \text{for } \alpha = 0. \end{cases}$$

Let $S = (S_m) m \in X$ be another commutative sequence of operators in H such that T_m and $S_{m'}$ commute for each m and m'. Denote by T^* the sequence $(T_m^*)_{m \in X}$ and by $T \cdot S$ the sequence $(T_m S_m)_{m \in X}$.

PROPOSITION 2. *The mapping $F_+(X) \ni \alpha \to \langle \alpha, T \rangle \in L(H)$ is a representation of the semi-group $F_+(X)$ and satisfies the conditions*

$$(9) \qquad \langle e_m, T \rangle = T_m, \quad m \in X;$$

$$(10) \qquad \langle \alpha, T^* \rangle = \langle \alpha, T \rangle^*;$$

$$(11) \qquad \langle \alpha, T \cdot S \rangle = \langle \alpha, T \rangle \langle \alpha, S \rangle.$$

If V stands for the sequence $(V_m)_{m\in X}$ as in Proposition 1, then the mapping $F_+(X)\ni\alpha\to\to\langle\alpha, V\rangle\in L(H_X)$ is an isometric representation of the semi-group $F_+(X)$ such that

(12) *H_X is the smallest subspace which contains $\bigcup_\alpha\langle\alpha, V\rangle\hat{H}$, where $\hat{H}=\{h\in H_X: h_\alpha=0$ for all $\alpha\neq 0\}$ and $\hat{H}$ is unitarily equivalent to H.*

Proof. (9), (10), (11) are clear. We prove (12). It is easy to see that the formula

$$(\langle\alpha, V\rangle h)_{\alpha'}=\begin{cases}h_{\alpha'-\alpha} & \text{for } \alpha'\geqslant\alpha;\\ 0 & \text{for other } \alpha'\end{cases} \tag{13}$$

holds true. Let $\hat{f}\,(f\in H)$ be the vector in $\hat{H}$ defined by

$$(\hat{f})_\alpha=\begin{cases}f & \text{for } \alpha=0,\\ 0 & \text{for } \alpha\neq 0.\end{cases}$$

Take $h=\hat{f}$ in (13). Then

$$(\langle\alpha, V\rangle\hat{f})_{\alpha'}=\begin{cases}f & \text{for } \alpha'=\alpha,\\ 0 & \text{for } \alpha'\neq\alpha.\end{cases}$$

This proves (12). Finally, the linear mapping $H\ni f\to\hat{f}\in\hat{H}$ is evidently unitary.

In what follows, $|\boldsymbol{T}|=(\Sigma_\alpha\|\langle\alpha, \boldsymbol{T}\rangle\|^2)^{1/2}$, where $\boldsymbol{T}=(T_m)_{m\in X}$ is a commutative sequence of operators in H. In case $\boldsymbol{T}=\{T\}$ we write $|T|=|\boldsymbol{T}|$.

3. Main results. Let H, K be Hilbert spaces. We say that a commutative sequence $\boldsymbol{T}=(T_m)_{m\in X}$ of operators in H is similar to a commutative sequence $\boldsymbol{S}=(S_m)_{m\in X}$ of operators in K, if there is a linear homeomorphism $U: H\to K$ such that

$$UT_m=S_mU \quad \text{for all } m\in X.$$

We then write $\boldsymbol{T}\underset{U}{\sim}\boldsymbol{S}$ or $\boldsymbol{T}\sim\boldsymbol{S}$. Notice that

PROPOSITION 3. *If $\boldsymbol{T}\underset{U}{\sim}\boldsymbol{S}$ then*

$$U\langle\alpha, \boldsymbol{T}\rangle=\langle\alpha, \boldsymbol{S}\rangle U \quad \text{for each } \alpha\in F_+(X), \tag{14}$$

$$\boldsymbol{T}^*\underset{W}{\sim}\boldsymbol{S}^*, \text{ where } W=(U^*)^{-1}. \tag{18}$$

We are now able to formulate our main result.

THEOREM 1. *Let $\boldsymbol{T}=(T_m)_{m\in X}$ be a commutative sequence of operators in a Hilbert space H. If $|\boldsymbol{T}|<+\infty$ then $\boldsymbol{T}$ is similar to a common part of the sequence $\boldsymbol{V}^*$,* i.e., *there is a closed subspace K of H_X such that each operator V_m^* leaves K invariant ($m\in X$), and $\boldsymbol{T}\sim\boldsymbol{S}=(S_m)_{m\in X}$ where $S_m=V_m^*|_K$.*

Proof. First we define the operator $W: H \to H_X$ by the formula

$$(Wh)_\alpha = \langle \alpha, T \rangle h \quad \text{for all } h \in H,\ \alpha \in F_+(X).$$

It is easy to check that the following inequalities hold

$$\|h\|^2 \leqslant \|Wh\|^2 \leqslant |T|^2 \|h\|^2 \quad \text{for all } h \in H. \tag{16}$$

Let $K = WH$. It follows from (16) that K is a closed subspace of H_X. Moreover, by Proposition 1, we have

$$WT_m h = \big((\langle \alpha + e_m, T \rangle h)_\alpha\big) = V_m^* Wh \quad \text{for all } h \in H,$$

hence

$$WT_m = V_m^* W. \tag{17}$$

This implies that each operator V_m^* leaves K invariant. Now define the operator $U: H \to K$, putting $Uh = Wh$ for $h \in H$. It follows from (16) and (17) that U is a linear homeomorphism such that

$$UT_m = V_m^*|_K U, \quad m \in X.$$

This completes the proof.

Recall [5] that a commutative sequence $\boldsymbol{S} = (S_m)_{m \in X}$ of operators in the Hilbert space H has a unitary dilation, if there is a Hilbert space $K \supset H$ and a commutative sequence $\boldsymbol{U} = (U_m)_{m \in X}$ of unitary operators in K such that

$$\langle \alpha, \boldsymbol{S} \rangle = P \langle \alpha, \boldsymbol{U} \rangle |_H, \quad \alpha \in F_+(X),$$

where P is the orthogonal projection of K onto H.

COROLLARY 1. *Let $\boldsymbol{T} = (T_m)_{m \in X}$ be a commutative sequence of operators in a Hilbert space H. If $|\boldsymbol{T}| < +\infty$ then there is a Hilbert space K and a commutative sequence $\boldsymbol{S} = (S_m)_{m \in X}$ of contractions in K such that $\boldsymbol{T} \sim \boldsymbol{S}$ and $\boldsymbol{S}$ has a unitary dilation.*

Proof. Since $|\boldsymbol{T}^*| = |\boldsymbol{T}| < +\infty$, it follows from Theorem 1 that there is a subspace K of H_X such that each operator V_m^* leaves K invariant and $\boldsymbol{T}^* \sim \boldsymbol{S}' = (S'_m)_{m \in X}$, where $S'_m = V_m^*|_K$. Thus, by Proposition 3, we have $\boldsymbol{T} \sim \boldsymbol{S} = \boldsymbol{S}'^*$. The sequence $\boldsymbol{S} = (S_m)_{m \in X}$ consists of mutually commuting contractions $S_m = (V_m^*|_K)^* = PV_m|_K$, where P is the orthogonal projection of H_X onto K. By a simple computation we get

$$\langle \alpha, \boldsymbol{S} \rangle = \Big\{ \prod_{m:\, \alpha(m) \neq 0} (V_m^*|_K)^{\alpha(m)} \Big\}^* = \{\langle \alpha, \boldsymbol{V} \rangle^* |_K\}^* = P \langle \alpha, \boldsymbol{V} \rangle |_K,$$

hence

$$\langle \alpha, \boldsymbol{S} \rangle = P \langle \alpha, \boldsymbol{V} \rangle |_K \quad \text{for all } \alpha \in F_+(X).$$

To complete the proof we can apply the Brehmer—Itô Theorem [5, Proposition 6.2, p. 22]. However we shall present another proof, because our situation is much simpler than that of the Brehmer—Itô Theorem. To begin with, we define the unitary operator U_m in $l_H^2(F(X))$ by

$$(U_m h)_\alpha = h_{\alpha - e_m} \quad \text{for all} \quad h \in l_H^2(F(X)).$$

If we identify the space H_X with the subspace $\{h \in l_H^2(F(X)) \colon h_\alpha = 0 \text{ for each } \alpha \notin F_+(X)\}$ of $l_H^2(F(X))$, then $U_m H_X \subset H_X$, and

$$U_m h = V_m h \quad \text{for all } h \in H_X.$$

The proof is finished.

4. Sufficient conditions for similarity to a part of the sequence V^*. Let $T = (T_m)_{m \in X}$ be a commutative sequence of operators in a complex Hilbert space H. Now we shall prove a sufficient condition in order that $|T| < +\infty$. We start with the simplest situation $T = \{T\}$.

PROPOSITION 4. *If the spectral radius of the operator T in H is strictly less than* 1, *then $|T| < +\infty$.*

Proof. It follows from the spectral radius formula [3, Theorem 2.38, p. 45], that $|T|^2 = \sum_{k=0}^{\infty} \|T^k\|^2 < +\infty$.

Let now $T = \{T_1, \ldots, T_n\}$ be an arbitrary commutative sequence of operators in H. It is not difficult to show that

$$\max_{1 \leq m \leq n} |T_m|^2 \leq |T|^2 \leq \prod_{m=1}^{n} |T_m|^2. \tag{18}$$

PROPOSITION 5. *If the spectral radius of all the operators T_m $(m = 1, 2, \ldots, n)$ is strictly less than* 1, *then $|T| < +\infty$.*

Consider now the case $T = (T_m)_{m=1}^{\infty}$. By standard arguments we can prove

$$|T| = \lim_{n \to \infty} |T_n|, \qquad T_n = (T_m)_{m=1}^{n}. \tag{19}$$

(18) and (19) imply

PROPOSITION 6. *If* $\sum_{k=1}^{\infty} \ln |T_k| < +\infty$ *then* $|T| < +\infty$.

Now we are able to prove

PROPOSITION 7. *If there is a positive integer m such that*

(20) *the spectral radius of T_k is strictly less than* 1 *for all* $k = 1, 2, \ldots, m$,

(21) $$\|T_k\| < 1 \quad \text{for all } k > m,$$

(22) $$\sum_{k=1}^{\infty} \|T_k\|^2 < +\infty,$$

then $|\boldsymbol{T}| < +\infty$.

Proof. It follows from (20), (18) and Proposition 4 that $\sum_{k=1}^{m} \ln |T_k| < +\infty$. Thus, by Proposition 6, it is enough to show that $\sum_{k>m} \ln |T_k| < +\infty$. We may assume that $\|T_k\|^2 \leqslant 1/2$ for all $k > m$ without loss of generality. Since

$$-\ln(1-x) \leqslant \frac{x}{1-x} \quad \text{for all} \quad x: 0 \leqslant x < 1,$$

and

$$\frac{x}{1-x} \leqslant 2x \quad \text{for all} \quad x: 0 \leqslant x \leqslant 1/2,$$

we get

$$\sum_{k>m} \ln |T_k| \leqslant 1/2 \sum_{k>m} \ln \frac{1}{1-\|T_k\|^2} \leqslant 1/2 \sum_{k>m} (-(\ln(1-\|T_k\|^2)) \leqslant$$

$$\leqslant 1/2 \sum_{k>m} \frac{\|T_k\|^2}{1-\|T_k\|^2} \leqslant \sum_{k>m} \|T_k\|^2 < +\infty.$$

This completes the proof.

5. Final remarks. It follows from Proposition 7 and Theorem 1 that each sequence $\boldsymbol{T} = (T_m)_{m=1}^{\infty}$ satisfying the conditions (20), (21) and (22) is similar to a common part of the sequence $\boldsymbol{V}^*$ (as in Proposition 1). In other words, the sequence $\boldsymbol{V}^*$ may be treated as a "universal model" for such families of operators. We notice that in the case when $X = \{1\}$ our model coincides with Rota's. Moreover, Proposition 4 and Theorem 1 imply the Rota theorem.

It is well know [5, Prop. 9.2, p. 39] that if a commutative sequence of contractions $\boldsymbol{T} = (T_m)_{m=1}^{\infty}$ satisfies the condition

$$\sum_{k=1}^{\infty} \|T_k\|^2 \leqslant 1,$$

then it has a unitary dilation. We can obtain a similar result from Proposition 7 and Corollary 1, i.e. if a sequence $\boldsymbol{T} = (T_m)_{m=1}^{\infty}$ satisfies the conditions (20), (21) and (22), then $\boldsymbol{T}$ is similar to a commutative sequence of contractions, which has a unitary dilation.

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