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On the Functional Inequality
$\psi(x+y) \leqslant \psi(x)\,\psi(y)$

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Many authors have dealt with subadditive functions (see e.g. [1], ch. VII) and we may say that subadditive functions are well known. The aim of the present paper is to give a form of general and general continuous solution of the inequality

$$\psi(x+y) \leqslant \psi(x)\psi(y)\,, \tag{1}$$

where $\psi\colon R \to R$ and R is the real line. This inequality has some application in tournament theory and for natural argument was dealt with by D. R. Snow [2].

First we are going to give two rather simple lemmas.

LEMMA 1. *If a function ψ fulfils inequality* (1) *and $\psi(x_0) = 0$ for an $x_0 \in R$, then*

$$\psi(x) \leqslant 0 \quad \textit{for } x \in R\,.$$

If ψ is a nonpositive function in R, then ψ fulfils inequality (1).

Proof. Let x be a real number. Putting in (1) x_0 in place of x and $x-x_0$ in place of y we have

$$\psi(x) \leqslant \psi(x_0)\psi(x-x_0) = 0\,.$$

The second part of the lemma is obvious.

LEMMA 2. *Positive solutions of inequality* (1) *are given by the formula*

$$\psi(x) = \exp\chi(x) \quad \textit{for } x \in R\,, \tag{2}$$

where χ is a subadditive function.

Proof. If ψ is a positive function, then inequality (1) is equivalent to the inequality

$$\chi(x+y) \leqslant \chi(x)+\chi(y) \quad \text{for } x \in R\,, \tag{3}$$

where

$$\chi(x) := \ln\psi(x) \quad \text{for } x \in R\,, \tag{4}$$

and then ψ is given by formula (2).

Remark. It follows from Lemma 1 that a continuous solution of inequality (1) may be either nonpositive or positive on R. In the first case it may be any continuous nonpositive function. In the second case it is given by formula (2) with an arbitrary function χ continuous and subadditive on R.

In the sequel we assume that a solution ψ of inequality (1) admits both positive and negative values (in this case ψ can be neither continuous nor vanishing at any point) and we define

$$A := \{x\colon\ \psi(x)<0\},$$
$$B := \{x\colon\ \psi(x)>0\}.$$

THEOREM 1. *If a function ψ is a solution of inequality* (1), $A, B \neq \emptyset$, *then* $\psi(0) = 1$ *and*$(B, +)$ *is a subgroup of* $(R, +)$.

Proof. First we shall prove that B is a group. Suppose $0 \in A$. Then for any $y \in B$ we would have $0<\psi(0+y)\leqslant\psi(0)\psi(y)<0$, which is impossible, and so $0 \in B$. Now for any $x \in B$, we have $0<\psi(0) = \psi(x-x) = \psi(x)\psi(-x)$ what, together with inequality $\psi(x)>0$, implies $-x \in B$. Suppose that $x+y \in A$ for some $x, y \in B$. Then

$$0<\psi(x) = \psi(x+y-y)\leqslant\psi(x+y)\psi(-y)<0,$$

which is impossible, and the proof that B is a group is completed.

Now we shall prove $\psi(0) = 1$. From (1) we have for $x = y = 0$

$$\psi(0) = \psi(0+0)\leqslant[\psi(0)]^2,$$

whence

$$\psi(0)(1-\psi(0))\leqslant 0.$$

The last inequality together with $\psi(0)>0$ implies that $\psi(0)\geqslant 1$. Suppose $\psi(0)>1$. Then for any $a \in A$ we have

$$\psi(a) = \psi(0+a)\leqslant\psi(0)\psi(a)<\psi(a),$$

which is impossible and the proof of the theorem is completed.

In the sequel $[a]$ will denote the coset generated by the element a and the group B. It is also convenient to denote by $\mathscr{A}$ a set which has exactly one common point with each sum of cosets $[a] \cup [-a]$, $[a] \in R/B$, $[a] \neq B$.

THEOREM 2. *If a function ψ satisfies inequality* (1) *and* $A, B \neq \emptyset$, *then there exist a subadditive function* $\chi\colon B \to R$ *and a family of functions* $\{\phi_a\}_{a\in\mathscr{A}}$, $\phi_a\colon [a] \cup [-a] \to R$ *such that*

$$\psi(x) = \begin{cases} \exp\chi(x) & \text{for } x \in B, \\ -\exp\phi_a(x) & \text{for } x \in [a] \cup [-a], \end{cases} \tag{4}$$

$$0\leqslant\phi_a(x)+\phi_a(-x) \quad \text{for } x \in R\setminus B,\ a \in \mathscr{A}, \tag{5}$$

$$\phi_a(x+y)\geqslant\phi_a(x)+\chi(y) \quad \text{for} \quad (x, y) \in [a]\times B. \tag{6}$$

Proof. Since $(B, +)$ forms a group (see Theorem 1), the following four cases are possible:

I. $x, y \in B$, $x+y \in B$,
II. $x, y \in A$, $x+y \in A$,
III. $x, y \in A$, $x+y \in B$,
IV. $x \in A$, $y \in B$, $x+y \in A$.

In the first case putting

$$\chi(x) := \ln \psi(x) \quad \text{for } x \in B \tag{7}$$

we get

$$\psi(x) = \exp \chi(x) \quad \text{for } x \in B$$

and we obtain from (7) and (1)

$$\chi(x+y) \leqslant \chi(x) + \chi(y) \quad \text{for } x, y \in B$$

what implies

$$\chi(0) \geqslant 0 . \tag{8}$$

The second case brings no restrictions on the function ψ.

In the third case x and y must be from opposite cosets, say $(x, y) \in [a] \times [-a]$ for an $a \in \mathscr{A}$. For these (x, y) let us write inequality (1) in the form

$$\psi(x+y) \leqslant (-\psi(x))(-\psi(y))$$

which yields, by virtue of (7)

$$\chi(x+y) \leqslant \ln(-\psi(x)) + \ln(-\psi(y)) . \tag{9}$$

Putting

$$\phi_a(x) := \ln(-\psi(x)) \quad \text{for } x \in [a] \cup [-a],\ a \in \mathscr{A} \tag{10}$$

and recalling $x+y \in B$ we see, by (9) and (8) that the functions ϕ_a fulfil inequality (5). Moreover, we have

$$\psi(x) = -\exp \phi_a(x) \quad \text{for } x \in [a] \cup [-a],\ a \in \mathscr{A} ,$$

as in (4).

In the fourth case x and $x+y$ are from the same coset, say $[a]$. Writing inequality (1) in the form

$$-\psi(x+y) \geqslant (-\psi(x))\psi(y)$$

we have

$$\ln(-\psi(x+y)) \geqslant \ln(-\psi(x)) + \ln \psi(y) \quad \text{for } (x, y) \in [a] \times B ,$$

whence, by virtue of (7) and (10), inequalities (6) are fulfilled and the proof of the theorem is completed.

THEOREM 3. *Let A' be a subset of R, $(B', +)$ — a subgroup of $(R, +)$, such that $R = A' \cup B'$, $A' \cap B' = \emptyset$, $A', B' \neq \emptyset$. Take a set $\mathscr{A}'$ which has exactly one common point* *with each sum of cosets* $[a] \cup [-a]$, $[a] \in R/B'$, $[a] \neq B]$ *and a family of functions* $\{\phi_a\}_{a \in \mathscr{A}'}$, $\phi_a\colon [a] \cup [-a] \to R$. *If a function* $\chi\colon B' \to R$ *is subadditive on* B' *and if inequalities* (5) *and* (6) (*with* $\mathscr{A}'$ *and* B' *in place of those* $\mathscr{A}$ *and* B, *respectively*) *are fulfilled, then the function* ψ *given by formula* (4) *fulfils inequality* (1).

Proof. First we prove that for $(x, y) \in [a] \times [-a]$ the following inequality

$$\chi(x+y) \leqslant \phi_a(x) + \phi_a(y) \tag{11}$$

is fulfilled. Let us take $x = a+b_1$, $y = -a+b_2$, $b_1, b_2 \in B'$. It follows from (6), (5) and subadditivity of χ that

$$\begin{aligned}\phi_a(x)+\phi_a(y) &= \phi_a(a+b_1)+\phi_a(-a+b_2) \geqslant \\ &\geqslant \phi_a(a)+\chi(b_1)+\phi_a(-a)+\chi(b_2) \geqslant \chi(b_1+b_2) = \chi(x+y)\end{aligned}$$

and then (11) holds.

Now, distinguishing cases I, II, III, and IV as in the proof of the Theorem 2 and using (6) and (11) we get (1) by a straightforward verification.

References

[1] E. Hille, R. S. Phillips, *Functional analysis and semi-groups*, (Colloq. Publ. Amer. Math. Soc., 31) Providence, Rhode Island, 1957.

[2] Donald R. Snow, *A functional inequality arising in combinatorics*, General Inequalities 2, 1980 Birkhäuser Verlag, Basel. Boston. Stuttgart., pp. 17-28.

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