ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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Strong Unicity in a Quotient Space

by Jan SUDOLSKI and Adam WÓJCIK

Abstract. Let f be an element of a normed linear space E and let g be its best approximation in an n-dimensional subspace V of E. We construct a quotient space $[E]$ such that the class $[g]$ is the strongly unique best approximation to $[f]$ in $[V]=\{[h]: h \in V\}$ with the preservation of distance and all best approximations to f in V are contained in $[g]$.

1. Introduction. Let E denote throughout the paper a normed linear space (real or complex) and V its n-dimensional subspace spanned by vectors $g_1, \ldots, g_n$.

An element $g \in V$ is called *the strongly unique best approximation to* $f \in E$ *in* V iff there exists a number $r>0$ such that for every $h \in V$

$$||f-h|| \geqslant ||f-g||+r||g-h|| .$$

By $P_V(f)$ we denote the set of all best approximations to f in V, i.e.

$$P_V(f) = \{g \in V: ||f-g|| \leqslant ||f-h||, h \in V\} .$$

The unit sphere in the space E^* of all continuous linear functionals on E is denoted by S^* and for every $f \in E$

$$M(f) := \{L \in S^*: Lf = ||f||\} .$$

The set of all extremal points of S^* is denoted by $\operatorname{Extr} S^*$ (i.e. $L \in \operatorname{Extr} S^*$ iff there is no closed line segment contained in S^* with L as an interior point).

The idea of the construction of the quotient space $[E]$ such that all best approximations to an element of E are contained in the same class is taken from Branningan [2], where the case of $E =$ the space of continuous functions is considered. By the way, a gap in the complex case of Theorem 4 in [2] is eliminated.

2. I-sets.

DEFINITION 2.1. Let $L_j \in \operatorname{Extr} S^*$ and let λ_j $(j = 1, \ldots, k)$ be such positive numbers, that $\sum_{j=1}^{k} \lambda_j = 1$. If for a basis $\{g_1, \ldots, g_n\}$ of V the equation

$$\begin{bmatrix} L_1(g_1), \ldots, L_k(g_1) \\ \cdots\cdots\cdots\cdots \\ L_1(g_n), \ldots, L_k(g_n) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \cdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}$$

is satisfied, then the system $[V, L_j, \lambda_j, k]$ is called *an I-set*.

If, moreover, for an $f \in E$ all $L_j \in M(f)$, then $[V, L_j, \lambda_j, k]$ is called *an I-set with respect to f*.

We note that an I-set does not depend on the choice of a basis for V because

Remark 2.2. A system $[V, L_j, \lambda_j, k]$ is an I-set if and only if

$$\sum_{j=1}^{k} \lambda_j L_j(h) = 0$$

for all $h \in V$.

The existence of I-sets in an arbitrary normed linear space E is guaranteed by the following theorem of Singer:

THEOREM 2.3 ([5], Theorem II.1.1). *Let $f \in E \setminus V$ and $g \in V$. Then g is a best approximation to f in V if and only if there exists (for some k) an I-set $[V, L_j, \lambda_j, k]$ with respect to $f-g$, where $1 \leqslant k \leqslant n+1$, if the scalars are real or $\alpha \leqslant k \leqslant 2n+1$, if the scalars are complex.*

By Remark 2.2 we immediately obtain

LEMMA 2.4. *If $[V, L_j, \lambda_j, k]$ is an I-set and for some element $h \in V$, $\mathrm{Re} L_j(h) \geqslant 0$ $(j = 1, \ldots, k)$, then $\mathrm{Re} L_j(h) = 0$ for all j.*

LEMMA 2.5. *If $[V, L_j, \lambda_j, k]$ is an I-set with respect to $f-g$, where g is a best approximation to f in V, then it is an I-set with respect to $f-p$, for every $p \in P_V(f)$.*

Proof. It is sufficient to show that $L_j \in M(f-p)$ for $j = 1, \ldots, k$. We have, for $j = 1, \ldots, k$,

$$\mathrm{Re} L_j(f-p) \leqslant |L_j(f-p)| \leqslant \|f-p\| = \|f-g\| = L_j(f-g)\,. \tag{2.1}$$

Hence $\mathrm{Re} L_j(p-g) \geqslant 0$. By Lemma 2.4 $\mathrm{Re} L_j(p-g) = 0$, what means that there is no strict inequality in (2.1). Thus $L_j(f-p)$ must be real and $L_j(f-p) = \|f-p\|$.

3. Main theorems. For an I-set $[V, L_j, \lambda_j, k]$ and $h \in E$ let $[h]$ be the class of equivalence of h,

$$[h] := \{\hat{h} \in E\colon\ L_j(h) = L_j(\hat{h}),\ j = 1, \ldots, k\}\,.$$

The set of all these classes is denoted by $[E]$ and will be considered as a normed vector space with the norm

$$\|[h]\|_0 := \max\{|L_j(h)| : j = 1, \ldots, k\}\,.$$

The subspace $[V] := \{[h]\colon h \in V\}$ has the same dimension as the rank of the matrix $[L_j(g_s)]$, where $\{g_1, \ldots, g_n\}$ is a basis of V.

THEOREM 3.1. *Let g be a best approximation to $f \in E \setminus V$ in V. If $[V, L_j, \lambda_j, k]$ is an I-set with respect to $f-g$, then $[g]$ is the unique best approximation to $[f]$ in $[V]$ and* $\mathrm{dist}([f], [V]) = \|f-g\|$.

Proof. Obviously $|||[f]-[g]|||_0 = ||f-g||$ since by the assumptions $L_j \in M(f-g)$. Suppose $h \in V$ is such that

$$|||[f]-[h]|||_0 \leqslant ||f-g||.$$

For every $j = 1, \ldots, k$ we have

$$L_j(f-g) = ||f-g|| \geqslant |L_j(f-h)| \geqslant \operatorname{Re} L_j(f-h).$$

Hence $\operatorname{Re} L_j(h-g) \geqslant 0$ and by Lemma 2.4,

$$\operatorname{Re} L_j(f-h) = |L_j(f-h)| = L_j(f-g).$$

Thus $L_j(f-h) = L_j(f-g)$, $j = 1, \ldots, k$, and consequently $[h] = [g]$.

Remark 3.2. There is a gap in Branningan's proof of a similar theorem (see [2], Theorem 4), namely the equality $\operatorname{Re}[zF(x)] = 0$, where z is a complex number different from zero and $F(x)$ is a value of a complex function F on x does not imply that $F(x) = 0$.

Remark 3.3. It is easy to see that all best approximations to f in V are contained in $[g]$. Indeed, if $\hat{g} \in P(f)$, then by Lemma 2.5, $L_j \in M(f-\hat{g})$ for every $j = 1, \ldots, k$. Hence

$$L_j(f-g) = ||f-g|| = ||f-\hat{g}|| = L_j(f-\hat{g}).$$

The following theorem is a generalization of results of Newman and Shapiro [4], Ault, Deutsch et al. [1] and Branningan [2]. Its proof is based on an idea of Newman and Shapiro.

THEOREM 3.4. *Let g be a best approximation to $f \in E \setminus V$ in V and $[V, L_j, \lambda_j, k]$ be an I-set with respect to $f-g$. There exists a positive constant r such that for every $h \in V$ the following inequality holds:*

$$|||[f]-[h]|||_0 \geqslant (|||[f]-[g]|||_0^2 + r|||[g]-[h]|||_0^2)^{1/2}$$

if the scalars are complex, or

$$|||[f]-[h]|||_0 \geqslant |||[f]-[g]|||_0 + r|||[g]-[h]|||_0$$

if the scalars are real.

Proof. Set $d = ||f-g|| = |||[f]-[g]|||_0$. For a fixed $h \in V$, put $m = |||[f]-[h]|||_0 - |||[f]-[g]|||_0 \geqslant 0$.

Hence for $j = 1, \ldots, k$, we have

$$|L_j(f) - L_j(h)| \leqslant d + m. \tag{3.1}$$

Thus

$$d^2 + 2d \operatorname{Re} L_j(g-h) + |L_j(g-h)|^2 \leqslant d^2 + 2dm + m^2,$$

whence

$$|L_j(g-h)|^2 \leqslant 2dm + m^2 - 2d \operatorname{Re} L_j(g-h). \tag{3.2}$$

Since $L_j \in M(f-g)$, from (3.1) we derive

$$\operatorname{Re} L_j(g-h) \leqslant m. \tag{3.3}$$

Put $p = \max\{\lambda_j^- : j = 1, \ldots, k\}$. Then $p>1$.

By Remark 2.2

$$\sum_{s=1}^{k} \lambda_s \operatorname{Re} L_s(g-h) = 0,$$

whence

$$-\operatorname{Re} L_j(g-h) = \sum_{s \neq j} \lambda_j^{-1} \lambda_s \operatorname{Re} L_s(g-h)$$

and by (3.3)

$$-\operatorname{Re} L_j(g-h) \leqslant pm. \tag{3.4}$$

Furthermore, by (3.2) and (3.4)

$$|L_j(g-h)|^2 \leqslant 2dm + m^2 + 2dpm.$$

Thus

$$|||[g]-[h]|||_0^2 \leqslant 2d(1+p)m + m^2.$$

Solving this inequality with respect to m, we get

$$m \geqslant [|||[g]-[h]|||_0^2 + d^2(1+p)^2]^{1/2} - d(1+p) \geqslant [(1+p)^{-2}|||[g]-[h]|||_0^2 + d^2]^{1/2} - d,$$

whence

$$m+d = |||[f]-[h]|||_0 \geqslant (|||[f]-[g]|||_0^2 + r|||[g]-[h]|||_0^2)^{1/2}$$

with $r = (1+p)^{-2}$.

In the real case (3.3) and (3.4) give

$$|L_j(g-h)| \leqslant pm.$$

From this inequality, putting $r = p^{-1}$ we derive

$$r|||[g]-[h]|||_0 \leqslant |||[f]-[h]|||_0 - |||[f]-[g]|||_0.$$

Corollary 3.5. *If in Theorem* 3.4 *the* I*-set is such that the matrix* $[L_j(g_s)]$ *is of rank* n, *then there exists* $c>0$ *such that for every* $h \in V$ *we have*

$$||f-h|| \geqslant (||f-g||^2 + c||h-g||^2)^{1/2},$$

if the scalars are complex, or

$$||f-h|| \geqslant ||f-g|| + c||h-g||,$$

i.e. g *is the strongly unique best approximation to* f *in* V, *if the scalars are real.*

Proof. Since the matrix $[L_j(g_s)]$ is of rank n, for every $h \in V$ we have $[h] \cap V = \{h\}$ and, consequently, for some $t>0$ independent of h

$$|||[h]-[g]|||_0 \geqslant t||h-g||,$$

since all norms on V are equivalent. Obviously

$$||f-g|| = |||[f]-[g]|||_0 \quad \text{and} \quad ||f-h|| \geqslant |||[f]-[h]|||_0 \quad \text{for each } h \in V,$$

so the corollary easily follows from Theorem 3.4.

We recall that V is said to be *a Chebyshev subspace of E*, if for each $f \in E$, there exists a unique best approximation to f in V.

Remark 3.6. In general the space $[V]$ need not be Chebyshev, e.g. consider $E = R^4$ with the norm

$$\|x\| = \max\{|x_1|, |x_2|, |x_3|\} + |x_4|$$

and $V = \operatorname{span}\{(1, 0, 0, 0), (0, 1, 0, 0)\}$.

Put $f = (0, 0, 0, 1)$. Then $g = (0, 0, 0, 0)$ is the strongly unique best approximation to f in V.

The space E^* may be identified with R^4 by the mapping

$$R^4 \ni (a_1, a_2, a_3, a_4) \to L \in E^*, \quad \text{where } L(x) = \sum_{j=1}^{4} a_j x_j .$$

Then

$$\|L\| = \max\{|a_1| + |a_2| + |a_3|, |a_4|\}$$

and

$$\operatorname{Extr} S^* = \{(k, 0, 0, l), (0, k, 0, l), (0, 0, k, l)\colon k, l = 1 \text{ or } -1\}$$

(consists of 12 points). By putting $k = 3$, $L_1 = (1, 0, 0, 1)$,

$$L_2 = (-1, 0, 0, 1),\ L_3 = (0, 0, 1, 1) \quad \text{and } \lambda_1 = \lambda_2 = \lambda_3 = \tfrac{1}{3}$$

we obtain an I-set with respect to $f - g$.

The mapping $T\colon [E] \ni [h] \to (z_1, z_2, z_3) \in R^3$, where $z_j = L_j(h)$ is an isometry between $[E]$ and R^3 endowed with the norm $\|z\| = \max\{|z_1|, |z_2|, |z_3|\}$. It is easy to see, that $T([V]) = \{z_1 + z_2 = z_3 = 0\}$. Since

$$P_{T([V])}((0, 0, 1)) = \{z_1 + z_2 = z_3 = 0,\ z_1 \in [-1, 1]\}$$

$[V]$ is not a Chebyshev subspace of $[E]$.

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