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Regular Trajectories of Closed Dynamical Systems in the Plane

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Introduction. We shall study closed dynamical systems in the plane. Beck in [1] gives a topological characterization of the sets which consist of critical points and describes the dynamical systems in the plane with compact trajectories. Knight in [6] classifies the structure of dynamical systems in terms of critical points. Mc Cann in [9] characterizes dynamical systems in the plane without critical points. The purpose of the present paper is to characterize the regular trajectories of closed dynamical systems in the plane.

The dynamical system is closed if each trajectory of this system is a closed set. Among these trajectories we mark regular trajectories (the trajectory $\pi(x)$ is regular if motion π^x: $R \ni t \to \pi(t, x) \in R^2$ is injective). Each such trajectory decomposes R^2 into two components (Remark 1). The main theorems of this paper are Theorems 3 and 4 which describe the behaviour of a dynamical system in a neighbourhood of a regular trajectory. With regard to this behaviour we may divide the set of regular trajectories into three disjoint subsets. Theorem 3 and 4 show all possibilities of behaviour of a closed dynamical system in the plane near given regular trajectory. They also enable us to prove some following corollaries about these systems (Section 3) and to circumscribe the prolongation and the prolongational limit set of a regular point (Section 4).

Let us note that some results of this paper are given by Knight [6]; in our note however we present them with different proofs. In particular, this is the case of Lemma 1, Remark 2, Proposition 8 and Corollary 2. Furthermore, Knight investigated closed dynamical systems paying attention to the structure of the set of critical points.

1. Throughout this paper R denotes the set of real numbers, R^n the Euclidean n-space, $I = \{x \in R\colon 0 \leqslant x \leqslant 1\}$, $S^n = \{x \in R^{n+1}\colon \sum_{i=1}^{n+1} x_i^2 = 1\}$ and $B^n = \{x \in R^n\colon \sum_{i=1}^{n} x_i^2 \leqslant 1\}$. For given set $M \subset R^n$, ∂M, $\operatorname{int} M$ and $\overline{M}$ denote the boundary, interior and closure of M, respectively. In the sequel (X, π) will denote the *dynamical system*, i.e. a topological space X with a continuous mapping $\pi\colon R \times X \to X$ satisfying $\pi(0, x) = x$ and $\pi(t, \pi(s, x)) = \pi(t+s, x)$ for any $x \in X$ and $t, s \in R$. We will denote the homeomorphism $X \ni x \to \pi(t, x) \in X$ by π_t. For any $M \subset X$ and $t \in R$ we define $\pi(t, M) = \{\pi(t, x)\colon x \in M\} =$

$= \pi_t(M)$. For every $x \in X$ we define:

$$\pi^+(x) = \pi(R_+ \times \{x\})$$

$$L^+(x) = \{y \in X\colon \text{ there exists sequence } \{t_n\} \subset R \text{ such that } t_n \to \infty,\ \pi(t_n, x \to y\}$$

$$J^+(x) = \{y \in X\colon \text{ there exist sequences } \{x_n\} \subset X,\ \{t_n\} \subset R \text{ such that } t_n \to \infty,\ x_n \to x,\ \pi(t_n, x_n) \to y\}$$

$$D^+(x) = \pi^+(x) \cup J^+(x).$$

By obvious modifications of the above we get symmetric sets: $\pi^-(x)$, $L^-(x)$, $J^-(x)$ and $D^-(x)$ (see for instance [3] or [4]). The sets $\pi(x)$, $L(x)$, $J(x)$ and $D(x)$ are defined as the unions of the positive and negative versions and are called the *trajectory*, the *limit set*, the *prolongational limit set* and the *prolongation* of x, respectively.

The *positive region of attraction* for a set $M \subset X$, i.e. $\{y \in X\colon \emptyset \neq L^+(y) \subset M\}$ will be denoted by $A^+(M)$; in the same way we define $A^-(M)$.

We say that $M \subset X$ is an *invariant set* if $\pi(x) \subset M$ for each $x \in M$. A set $M \subset X$ is said to be *positively* (*negatively*) *stable* if for every neighbourhood U of M and every $x \in M$ there is a neighbourhood V of x such that $\pi(t, V) \subset U$ whenever $t \geqslant 0$ ($t \leqslant 0$). A point $x \in X$ is said to be *positively* (*negatively*) *Poisson stable* if $x \in L^+(x)$ $(x \in L^-(x))$. We shall denote the sets of *critical* (i.e. $\pi(x) = \{x\}$), *periodic* (i.e. there exists $t \neq 0$ such that $\pi(t, x) = x$, but $\pi(x) \neq \{x\}$) and *regular* (i.e. the function $R \ni t \to \pi(t, x) \in X$ is injective) points by S, P and T, respectively. A dynamical system (X, π) is said to be a *closed dynamical system* if $\pi(x)$ is a closed subset of X for each $x \in X$.

By a *simple arc* or *simple closed curve* we mean a subset C of X homeomorphic to I or S^1, respectively. If X is a 2-manifold, then a simple arc $C \subset X$ is said to be a *transversal arc* if and only if there exists $\varepsilon > 0$ such that $\pi(t_1, C) \cap \pi(t_2, C) = \emptyset$ for every $0 \leqslant t_1 < t_2 \leqslant \varepsilon$.

We say that X is a *dichotomic space* (see [5]) if it is an orientable 2-manifold satisfying the Jordan curve separation property i.e. if every simple closed curve C in X decomposes X into two connected open sets G_1, G_2 with common boundary C. From Jordan Theorem R^2 is a dichotomic space; in this case we define In C as a bounded component of $R^2 \setminus C$.

The following known propositions will be used in the sequel (see for example [5], [4], [2]).

PROPOSITION 1 (VII.2 in [5]). *If X is a 2-manifold, then*

(i) *for every $x \in X \setminus S$ there exists a transversal arc which contains x as a non-end-point*

and

(ii) *if C is a transversal arc then for sufficiently small positive ε the set* $\bigcup_{-\varepsilon < t < \varepsilon} \pi(t, C)$ *is a neighbourhood of all non-end-points of C.*

PROPOSITION 2 (VII.4 in [5]). *If X is a dichotomic space then*

(i) *every periodic trajectory meets any transversal arc at one point at most*

(ii) *if a transversal arc C and a trajectorial arc L form a simple closed curve then one component of* $X\setminus(C \cup L)$ *is positively invariant, the second is negatively invariant.*

PROPOSITION 3 (VIII.1 in [5]). *If X is a dichotomic space and x is a positively (or negatively) Poisson stable point then x is a critical or periodic point.*

PROPOSITION 4 (Cor. VI.1.2 in [4]). *Let X be a locally compact space, M a compact invariant subset of X. Then one of the conditions* (C.1)–(C.4) *holds.*

(C.1) *M is positively asymptotically stable, i.e. M is positively stable and* $A^+(M)$ *is a neighbourhood of M.*

(C.2) *M is negatively asymptotically stable, i.e. M is negatively stable and* $A^-(M)$ *is a neighbourhood of M.*

(C.3) *there exist* $x \notin M$, $y \notin M$ *such that* $\emptyset \neq L^+(x) \subset M$ *and* $\emptyset \neq L^-(y) \subset M$.

(C.4) *every neighbourhood U of M contains an* $x \notin M$ *with* $\pi(x) \subset U$.

PROPOSITION 5 (V.3.8 in [4]). *If* $X = R^n$ *and M is a compact positively invariant subset of X homeomorphic to* B^n, *then M contains a critical points.*

PROPOSITION 6 (III.2.8 in [4]). *If X is a complete metric space then* $\overline{L^+(x)\setminus\pi(x)} = \overline{\pi(x)}$ *for every regular positively Poisson stable point* $x \in X$ *(similarly for negatively Poisson stable point).*

We will also use the following

Schönflies Theorem (see for example Cor. 22 in [2]). *Every homeomorphism from* S^1 *into* R^2 *can be extended to a homeomorphism of* R^2 *onto itself.*

Now we give two lemmas which will be used later. (X, π) denotes a dynamical system on a topological space X.

LEMMA 1. *Let X be a complete metric space and let* $x \in X$ *be a regular point. If* $\pi(x)$ *is a closed subset of X then* $L^+(x) = L^-(x) = \emptyset$.

Proof. We have $L^+(x) \subset \overline{\pi(x)} = \pi(x)$. Suppose that $y \in L^+(x)$ for some $y \in X$. Thus $y \in \pi(x)$ and $\pi(y) = \pi(x)$. From the invariantness of $L^+(x)$ we obtain $\pi(y) \subset L^+(x) \subset \pi(x)$, hence $L^+(x) = \pi(x)$. By using Proposition 5 we have $\overline{L^+(x)\setminus\pi(x)} = \overline{\pi(x)} = \pi(x)$ that implies $\pi(x) = \emptyset$. This is not possible. The proof $L^-(x) = \emptyset$ is similar.

Remark 1. It is obvious that if $x \in R^2$ is such that $L(x) = \emptyset$ then $\pi(x)$ decomposes R^2 into two connected open unbounded sets with common boundary $\pi(x)$. Therefore by Lemma 1 every regular trajectory in the closed dynamical system on the plane decomposes R^2 into such sets.

LEMMA 2. *Let X be a dichotomic space. If C is a transversal arc and N is a trajectory, then every point of the set* $N \cap C$ *is isolated in C with the induced topology.*

Proof. It is obvious that $N \cap C = \emptyset$ if N is a critical point. Proposition 2(i) implies that $N \cap C$ has at most one point when N is a periodic trajectory. It remains to show that lemma holds if N is a regular trajectory. Let $x \in N \cap C$ and let $f\colon [-1, 1] \to C$ be a homeomorphism. We have $f(t_0) = x$ for some $t_0 \in [-1, 1]$. In order to finish the proof we shall show that there exists $\delta \in (0, 1]$ such that

$$f([-1, 1] \cap [t_0-\delta, t_0+\delta]) \cap \pi(x) = \{x\}.$$

Assume the contrary: that for every natural number $n \geq 1$ there is $s_n \neq 0$ and

$$t_n \in \left[t_0 - \frac{1}{n},\ t_0 + \frac{1}{n}\right] \cap [-1, 1]$$

such that $\pi(s_n, x) = f(t_n)$. By virtue of Proposition 3 we obtain $x \notin L(x)$, therefore $\{s_n\}$ is a bounded sequence. Thus there exists a subsequence $\{s_{n_k}\}$ convergent to certain $s_0 \in R$. We will show that $s_0 = 0$. Indeed, $\pi(s_{n_k}, x) \to \pi(s_0, x)$ and $\pi(s_{n_k}, x) = f(t_{n_k}) \to f(t_0) = x$. Since x is a regular point we have $s_0 = 0$. Now take $0 < \varepsilon < 1$ for C from the definition of a transversal. Then there is s_{n_l} such that $|s_{n_l}| < \varepsilon$. Let for example $s_{n_l} > 0$ (the case $s_{n_l} < 0$ is similar). Hence $f(t_{n_l}) = \pi(s_{n_l}, x) \in C \cap \pi(s_{n_l}, C)$. This contradiction completes the proof of Lemma 2.

2. Throughout this section (R^2, π) denotes a closed dynamical system in the plane. Let $x, y \in R^2$ be different periodic points. Then we say that $\pi(y)$ *surrounds* $\pi(x)$ if $\operatorname{In}\pi(x) \subset \operatorname{In}\pi(y)$ (definition of $\operatorname{In} C$, where C is a simple closed curve was given in Section 1).

THEOREM 1. *If $x \in P$ then there exists $y \in P$ such $\pi(y)$ surrounds $\pi(x)$ and*

$$\operatorname{In}\pi(y) \setminus \operatorname{In}\pi(x) \subset P.$$

Proof. Let f be a homeomorphism from R^2 onto R^2 such that $f(\pi(x)) = S^1$ which exists by Schönflies Theorem. Then (R^2, π_f) is a closed dynamical system with $\pi_f(s, z) = f(\pi(s, f^{-1}(z)))$ for every $s \in R$ and $z \in R^2$. Restrict this dynamical system to the invariant subset $R^2 \setminus \operatorname{In} S^1$. We will use Proposition 4 with $X = R^2 \setminus \operatorname{In} S^1$ and $M = S^1$. It follows from Lemma 1 that $L^+(z) \cap S^1 = \emptyset$ and $L^-(z) \cap S^1 = \emptyset$ for every $z \notin S^1$. Thus the condition (C.3) does not hold. Similarly $A^+(S^1) = A^-(S^1) = S^1$ which implies that (C.1) and (C.2) also do not hold. Therefore by Proposition 4 we obtain that every neighbourhood U of S^1 contains a point $u \notin S^1$ with $\pi_f(u) \subset U$. Since S is a closed set one can find $\varepsilon > 0$ such that the neighbourhood $U_\varepsilon = \{y \in R^2\colon 1 \leq |y| < 1+\varepsilon\}$ of S^1 contains no critical point. There exists a point $u \in U_\varepsilon$ such that $\pi_f(u) \subset U_\varepsilon$. As U_ε is a bounded set, it follows from Lemma 1 that the set U_ε contains only periodic points. Well known properties of Jordan curves in the plane imply that exactly one of the following inclusions is satisfied

(i) $\operatorname{In}\pi_f(u) \subset U_\varepsilon$

or

(ii) $\operatorname{In}\pi_f(u) \supset \operatorname{In} S^1$.

In virtue of Proposition 5 the case (i) is not possible as U_ε contains no critical point. Thus (ii) is fulfilled, hence $\pi_f(u)$ surrounds $S^1 = \pi_f(x)$ and $\mathrm{In}\,\pi_f(u)\setminus\mathrm{In}\,\pi_f(x)\subset P$, obviously. Using elementary properties of a homeomorphism f we show that the point $y = f^{-1}(u)$ satisfies the assertion of the theorem. This completes the proof.

Remark 2. We can repeat this proof and show that there exists a periodic point $y\in\mathrm{In}\,\pi(x)$ such that $\mathrm{In}\,\pi(x)\setminus\mathrm{In}\,\pi(y)\subset P$. Now it is easy to show by using previous theorem that P is an open set. Taking sufficiently small $\varepsilon>0$ in the last proof we can show stability of $\pi(x)$. Then we can get another way the results obtained by Knight in [6].

THEOREM 2. *Every trajectory meets any transversal arc at one point at most.*

Proof. Let $N = \pi(x)$ be a trajectory of a point $x\in R^2$ and C a transversal arc such that $x\in C$. It is clear (by Proposition 2 (i)) that one can assume that x is a regular point. Suppose the contrary: $(N\setminus\{x\})\cap C\neq\emptyset$. Then Lemma 2 implies that there exists $z\in C\cap N$ such that subarc $\overset{\frown}{x\,z}$ of C contains no point of the trajectory N. Let $\overline{x\,z}$ denote the segment of the trajectory N from x to z. Hence the subarc $\overset{\frown}{x\,z}$ and the segment $\overline{x\,z}$ form a single curve K. It follows from Proposition 2 (ii) that the set $\mathrm{In}\,K$ is either positively or negatively invariant. The set $\mathrm{In}\,K$ is bounded, so by Lemma 1 its every point is either periodic or critical. Thus $\mathrm{In}\,K$ and consequently, $\overline{\mathrm{In}\,K}$ are invariant. Since $x\in\overline{\mathrm{In}\,K}$ so $N = \pi(x)\subset\overline{\mathrm{In}\,K}$. It is not possible because N is a regular trajectory and $\overline{\mathrm{In}\,K}$ is bounded.

As a simple consequence of the previous theorem we obtain the following

COROLLARY 1. *Let $x\in R^2$ be a regular point and let C be a transversal arc such that x is a non-end-point of C. Assume that y is one of two end-points of C. By D denote this of two components of $R^2\setminus\pi(x)$ (see Remark 1) which contains the point y. Then*

(a) *the subarc $\overset{\frown}{y\,x}$ without the point x is contained in D*

and

(b) *the set $\bigcup_{t\in R}\pi(t,\overset{\frown}{y\,x}\setminus\{y\})$ is an open, connected and invariant neighbourhood of the trajectory $\pi(x)$ in $\bar{D}$.*

Proof. The assertion (a) follows immediately from Theorem 2. Now we shall show that $A = \bigcup_{t\in R}\pi(t,\overset{\frown}{y\,x}\setminus\{y\})$ is open in D. Let z denote another end-point of C. Then by Proposition 1 (ii) for some $\varepsilon>0$ there exists the open set V such that

$$C\setminus\{y,z\}\subset V\subset\bigcup_{-\varepsilon<t<\varepsilon}\pi(t,C)$$

and V is disjoint with $\pi(y)\cup\pi(z)$. Thus A is open as $A = \bar{D}\cap\big(\bigcup_{t\in R}\pi_t(V)\big)$ and π_t is a homeomorphism. Since $W_t = \pi(t,\overset{\frown}{y\,x}\setminus\{y\})\cup\pi(x)$ is a connected set containing $\pi(x)$ for each $t\in R$, so the set $A = \bigcup_{t\in R}W_t$ is connected. It is obvious that A is invariant.

We recall that T denotes the set of all regular points.

PROPOSITION 7. *If C is a transversal arc then the set $C\cap T$ form a subarc of C.*

Proof. In order to prove the proposition it is sufficient to show that $C \cap T$ is a closed set and $\overset{\frown}{x_1x_2}\subset T$ whenever $x_1, x_2 \in C \cap T$. The set $C \cap T$ is closed as P is an open set (see Remark 2). Now let $x_1, x_2 \in C \cap T$. We want to show that $\overset{\frown}{x_1x_2}\subset T$. Suppose that there is $y \in \overset{\frown}{x_1x_2}\setminus T$. Obviously y is not a critical point. Hence $y \in P$. By Theorem 1 there exists a periodic point z such that $\overline{\mathrm{In}\,\pi(y)}\subset \mathrm{In}\,\pi(z)$. The set $\overline{\mathrm{In}\,\pi(z)}$ is bounded and invariant what implies $x_1, x_2 \notin \overline{\mathrm{In}\,\pi(z)}$. Since $y \in \mathrm{In}\,\pi(z)$, we have $\overset{\frown}{x_1y} \cap \pi(z) \neq \emptyset$ and $\overset{\frown}{y\,x_2} \cap \pi(z) \neq \emptyset$. Then the periodic trajectory $\pi(z)$ meets the transversal arc C more than once. By Proposition 2 (i) it is impossible. The proof of the proposition is completed.

Now we can state our main theorems.

THEOREM 3. *Let $x \in R^2$ be a regular point and let D be one of the components of the set $R^2\setminus\pi(x)$. Then there exists an open invariant connected neighbourhood of the trajectory $\pi(x)$ in $\bar{D}$ such that $U\setminus\pi(x)\subset P$ or $U\subset T$.*

Proof. Applying Proposition 1 (i) we get that there is a transversal arc C which contains x as a non-end-point. Let y be an end-point of C contained in D and z another end-point of C. If the set $\overset{\frown}{x\,y}\setminus\{x\}$ contains any regular point w then using Corollary 1 for the transversal arc $\overset{\frown}{x\,w}$ and Proposition 7 we obtain U satisfying the assertion of the theorem with $U\subset T$. If the set $\overset{\frown}{x\,y}\setminus\{x\}$ contains no regular points then in the same way we obtain U satisfying the assertion of the theorem with $U\subset P$. This completes the proof of the theorem.

In order to state and prove our next theorem, we will first recall some definitions and propositions. By S_∞ is denoted the union of unbounded components of S.

DEFINITION. We denote $N_x = \bigcup \{\mathrm{In}\,\pi(y)\colon\ x \in \mathrm{In}\,\pi(y)\}$

Remark 3. Theorem 1 implies that for every periodic point x the set N_x is not empty.

DEFINITION. A nonempty set A is called an *N-set* if there exists a point $x \in R^2$ such that $A = N_x$.

Remark 4. It is obvious that for every N-set A and for every regular point y there is $\pi(y) \cap A = \emptyset$.

PROPOSITION 8. *The family of N-sets is a finite or countable collection of open, connected, invariant and pairwise disjoint sets.*

This proposition is obvious; only the last part needs a proof which is immediate by the following lemma.

LEMMA 3. *If x, y are contained in any N-set, then there exists a periodic point z such that $\{x, y\}\subset \mathrm{In}\,\pi(z)$.*