

Weak Stability in Generalized Local Pseudo-dynamical Systems

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Abstract. The weak stability of sets in generalized local pseudo-dynamical systems is considered. Some theorems on connections between this stability and the existence of Lyapunov function are given.

Introduction. The objective of this paper is to give a dependence between so-called weak stability of a set and the existence of Lyapunov function. The idea introduced in [11] is used. It is some generalization of A. Pelczar's conception ([7], [8], [9]). In Section 1 we recall the definition of the generalized local pseudo-dynamical system and some basic definitions. In Section 2 we give the definition of the weak stability of a set. The conception to consider the weak stability of the sets was suggested by A. Pelczar. In the next section we introduce the definition of a weak Lyapunov function and we show that its existence implies the weak stability. In Section 4 we prove that this theorem can be inverted for some generalized local pseudo-dynamical systems on the plane. We give an application in the next section. The author is deeply indebted to Professor A. Pelczar for his remarks when writing this paper.

1. We will use here the notation and definitions from [11]. The set of real numbers (real non-negative) will be denoted by $R(R^*)$. We recall the definition of a generalized local pseudo-dynamical system.

Let X be a non-empty set, $(U, +)$ be an abelian semi-group with the neutral element 0. Let us suppose that A is a subset of $U \times X$ such that $\{0\} \times X \subset A$. We put

$$I_x = \{t \in U: (t, x) \in A\}$$

for every $x \in X$. Let $\mathcal{P}(X)$ denote the family of non-empty subset of X and let λ be a mapping from A into $\mathcal{P}(X)$. By a *generalized local pseudo-dynamical system* we mean a quadruplet (X, U, A, λ) such that

- (i) $\lambda(0, x) = \{x\}$ for every $x \in X$,
- (ii) $s+t \in I_x$ for every $s \in I_x, y \in \lambda(s, x), t \in I_y$,
- (iii) $\lambda(t, y) \subset \lambda(t+s, x)$ for every $s \in I_x, y \in \lambda(s, x), t \in I_y$,

We define $\lambda_t(x) = \lambda(t, x)$, $\lambda(x) = \bigcup \{\lambda_s(x) : s \in I_x\}$, $\lambda(B) = \bigcup \{\lambda(y) : y \in B\}$ for $x \in X$, $t \in I_x$, $B \subset X$. The set $\lambda(x)$ is called the trajectory of the point x . A set $M \subset X$ is said to be *weak invariant* if and only if $\lambda_t(x) \cap M \neq \emptyset$ for every $x \in M$ and $t \in I_x$ (we recall that M is an *invariant set* iff $\lambda_t(x) \subset M$ for every $x \in M$ and $t \in I_x$). It is obvious that every invariant set is weak invariant.

2. Suppose that (X, U, A, λ) is the generalized local pseudo-dynamical system. Let $\beta: X \rightarrow \mathcal{P}(\mathcal{P}(X))$ be a mapping and let Ω be a non-empty subfamily of $\mathcal{P}(X)$. Let us assume that M is a non-empty subset of X . In the following $\tilde{\beta}$ will denote the family $\bigcup \{\beta(x) : x \in X\}$. We consider the following table (see [11]).

variable	domain	existential quantifier	universal quantifier
ω	Ω	o	O
ζ	M	z	Z
δ	$\tilde{\beta}$	d	D
η	X	h	H

The table contains the variables, their domains and the quantifiers. Every variable will be denoted by a Greek letter and the corresponding universal (or existential) quantifier by the corresponding roman upper-case or lower-case letter, respectively. Let σ represent any of the variables listed in the table. Then $\underset{\sigma}{\supset}$ denotes \wedge (conjunction) if it occurs within the scope of the corresponding existential quantifier and denotes \Rightarrow (implication) if it occurs within the scope of the corresponding universal quantifier. Let the symbol Σ represent the proposition

$$\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} \Lambda(\omega, \eta)],$$

where $\Lambda(\omega, \eta)$ is the following propositional function

$$\forall s \in I_y, \quad \lambda_s(\eta) \cap \omega \neq \emptyset$$

of the two variables ω, η . Note that Σ is a propositional function of the four variables $\delta, \zeta, \omega, \eta$.

Any sequence of the letters from the third and fourth columns of the table will be called a *word*. We say that the word is *feasible* if it contains only one letter from each row. The class of feasible words will be denoted by \mathcal{W} . Let $S \in \mathcal{W}$. Let replace the letters of S by the quantifiers. If we put this sequence of quantifiers before the propositional function Σ we obtain a proposition without free variables. This proposition will be denoted by $S\Sigma$.

DEFINITION 1. The set M is said to be (S, Ω, β) -*weak stable* if and only if the proposition $S\Sigma$ is true.