

Weak Stability in Generalized Local Pseudo-dynamical Systems

by Andrzej TRZEPIZUR

Abstract. The weak stability of sets in generalized local pseudo-dynamical systems is considered. Some theorems on connections between this stability and the existence of Lyapunov function are given.

Introduction. The objective of this paper is to give a dependence between so-called weak stability of a set and the existence of Lyapunov function. The idea introduced in [11] is used. It is some generalization of A. Pelczar's conception ([7], [8], [9]). In Section 1 we recall the definition of the generalized local pseudo-dynamical system and some basic definitions. In Section 2 we give the definition of the weak stability of a set. The conception to consider the weak stability of the sets was suggested by A. Pelczar. In the next section we introduce the definition of a weak Lyapunov function and we show that its existence implies the weak stability. In Section 4 we prove that this theorem can be inverted for some generalized local pseudo-dynamical systems on the plane. We give an application in the next section. The author is deeply indebted to Professor A. Pelczar for his remarks when writing this paper.

1. We will use here the notation and definitions from [11]. The set of real numbers (real non-negative) will be denoted by $R(R^*)$. We recall the definition of a generalized local pseudo-dynamical system.

Let X be a non-empty set, $(U, +)$ be an abelian semi-group with the neutral element 0. Let us suppose that A is a subset of $U \times X$ such that $\{0\} \times X \subset A$. We put

$$I_x = \{t \in U: (t, x) \in A\}$$

for every $x \in X$. Let $\mathcal{P}(X)$ denote the family of non-empty subset of X and let λ be a mapping from A into $\mathcal{P}(X)$. By a *generalized local pseudo-dynamical system* we mean a quadruplet (X, U, A, λ) such that

- (i) $\lambda(0, x) = \{x\}$ for every $x \in X$,
- (ii) $s+t \in I_x$ for every $s \in I_x, y \in \lambda(s, x), t \in I_y$,
- (iii) $\lambda(t, y) \subset \lambda(t+s, x)$ for every $s \in I_x, y \in \lambda(s, x), t \in I_y$,

We define $\lambda_t(x) = \lambda(t, x)$, $\lambda(x) = \bigcup \{\lambda_s(x) : s \in I_x\}$, $\lambda(B) = \bigcup \{\lambda(y) : y \in B\}$ for $x \in X$, $t \in I_x$, $B \subset X$. The set $\lambda(x)$ is called the trajectory of the point x . A set $M \subset X$ is said to be *weak invariant* if and only if $\lambda_t(x) \cap M \neq \emptyset$ for every $x \in M$ and $t \in I_x$ (we recall that M is an *invariant set* iff $\lambda_t(x) \subset M$ for every $x \in M$ and $t \in I_x$). It is obvious that every invariant set is weak invariant.

2. Suppose that (X, U, A, λ) is the generalized local pseudo-dynamical system. Let $\beta: X \rightarrow \mathcal{P}(\mathcal{P}(X))$ be a mapping and let Ω be a non-empty subfamily of $\mathcal{P}(X)$. Let us assume that M is a non-empty subset of X . In the following $\tilde{\beta}$ will denote the family $\bigcup \{\beta(x) : x \in X\}$. We consider the following table (see [11]).

variable	domain	existential quantifier	universal quantifier
ω	Ω	o	O
ζ	M	z	Z
δ	$\tilde{\beta}$	d	D
η	X	h	H

The table contains the variables, their domains and the quantifiers. Every variable will be denoted by a Greek letter and the corresponding universal (or existential) quantifier by the corresponding roman upper-case or lower-case letter, respectively. Let σ represent any of the variables listed in the table. Then $\underset{\sigma}{\supset}$ denotes \wedge (conjunction) if it occurs within the scope of the corresponding existential quantifier and denotes \Rightarrow (implication) if it occurs within the scope of the corresponding universal quantifier. Let the symbol Σ represent the proposition

$$\delta \in \beta(\zeta) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} \Lambda(\omega, \eta)],$$

where $\Lambda(\omega, \eta)$ is the following propositional function

$$\forall s \in I_y, \quad \lambda_s(\eta) \cap \omega \neq \emptyset$$

of the two variables ω, η . Note that Σ is a propositional function of the four variables $\delta, \zeta, \omega, \eta$.

Any sequence of the letters from the third and fourth columns of the table will be called a *word*. We say that the word is *feasible* if it contains only one letter from each row. The class of feasible words will be denoted by \mathcal{W} . Let $S \in \mathcal{W}$. Let replace the letters of S by the quantifiers. If we put this sequence of quantifiers before the propositional function Σ we obtain a proposition without free variables. This proposition will be denoted by $S\Sigma$.

DEFINITION 1. The set M is said to be (S, Ω, β) -*weak stable* if and only if the proposition $S\Sigma$ is true.

Let S be a feasible word. We will add one variable α , the domain of which will be $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{A} \neq \emptyset$. We shall denote the corresponding quantifiers by a, A in accordance with convention. If we replace in S the letter $o(O)$ by the letter $a(A)$ we obtain the word which will be denoted by S_1 . By S_2 we mean the following word

$$S_2 = \begin{cases} Oa, & \text{if } O \text{ occurs in the word } S \\ Ao, & \text{if } o \text{ occurs in the word } S. \end{cases}$$

Let us consider the formulas Q_1, Q_2, Q_3

$$Q_1: \delta \in \beta(\zeta) \underset{\delta}{\supset} (\eta \in \delta \underset{\eta}{\supset} \eta \in \alpha),$$

$$Q_2: \alpha \subset \omega,$$

$$Q_3: \forall \alpha \in \mathcal{A}, \alpha \text{ is a weak invariant set.}$$

THEOREM 1. *Let us suppose that there exists a non-empty family $\mathcal{A} \subset \mathcal{P}(X)$ such that*

(a) $S_1 Q_1$,

(b) $S_2 Q_2$,

(c) Q_3 .

Then M is (S, Ω, β) -weak stable.

Proof. Let the letter O occurs in the word S . Since the quantifiers in the front O are the same in the word S and S_1 , we make the same corresponding choices up to the position of O in the formula $S\Sigma$ as in the condition $S_1 Q_1$. Let now ω be an arbitrarily fixed element of Ω . From (b) we obtain that there exists $\alpha \in \mathcal{A}$ such that $\alpha \subset \omega$. We finish the condition $S_1 Q_1$ with this α and we obtain the whole choice in the formula $S\Sigma$. Finally we show that the choice is correct. From the condition (c) we have that α is a weak invariant set. Since $\eta \in \alpha$, we have $\forall s \in I_\eta, \lambda_s(\eta) \cap \alpha \neq \emptyset$. As $\alpha \subset \omega$, we obtain that the proposition $S\Sigma$ holds. The argumentation in the case o in the word S is similar. The proof of Theorem 1 is completed.

Remark 1. It is easy to see that the inverse theorem is not true.

3. Let (T, \leq) be a partially ordered space. We put

$$P = \begin{cases} T, & \text{if } \inf T \text{ does not exist,} \\ T \setminus \{\inf T\}, & \text{if } \inf T \text{ exists.} \end{cases}$$

DEFINITION 2. A function $V: W \rightarrow T$, where $W \subset X$, $W \neq \emptyset$ is said to be a (S, Ω, β) -weak Lyapunov function for M if and only if the family $\mathcal{A} = \{A_\xi\}_{\xi \in P}$, where

$$A_\xi = \{y \in W: V(y) \leq \xi\},$$

fulfils the conditions (a) and (b) from Theorem 1 and the condition

(i) for every $y \in W$ and $s \in I_y$, there exists $z \in \lambda_s(y)$ such that $z \in W$ and $V(z) \leq V(y)$.

THEOREM 2. *If there exists a (S, Ω, β) -weak Lyapunov function for M then M is (S, Ω, β) -weak stable.*

Proof. Let us first prove that $\mathcal{A} = \{A_\xi\}_{\xi \in P}$ fulfils the condition (c) from Theorem 1, i.e. that every $\alpha \in \mathcal{A}$ is a weak invariant set. Let $A_\xi \in \mathcal{A}$, $\xi \in P$ be arbitrarily fixed and let $y \in A_\xi$, $s \in I_y$. In virtue of (i) we have that there exists $z \in \lambda_s(y)$ such that $z \in W$ and $V(z) \leq V(y) \leq \xi$. Thus $z \in \lambda_s(y) \cap A_\xi$ and $\lambda_s(y) \cap A_\xi \neq \emptyset$. Hence the family \mathcal{A} fulfils the condition (c). From Definition 2 we obtain that \mathcal{A} fulfils (a) and (b). Theorem 1 implies immediately that the set M is (S, Ω, β) -weak stable set.

4. The inverse theorem is not true in general. In this section we show that it can be inverted for some class of the generalized local pseudo-dynamical systems on the plane.

Let us consider the pair (Φ, Ψ) of the families of continuous functions

$$\Phi = \{\varphi_{(t,x)}\}_{(t,x) \in R^2} \quad \text{and} \quad \Psi = \{\psi_{(t,x)}\}_{(t,x) \in R^2}$$

such that $\varphi_{(t,x)}, \psi_{(t,x)}$ are the mappings from $\Delta_{(t,x)}$ into R , where $\Delta_{(t,x)} \subset R$ is an interval of the form $[t, a)$ ($t < a \leq \infty$: a depends on t and x). We assume that $x = \varphi_{(t,x)}(t) = \psi_{(t,x)}(t)$ and $\varphi_{(t,x)} \leq \psi_{(t,x)}$ for each $(t, x) \in R^2$ and

$$\varphi_{(t+u, \varphi_{(t,x)}(t+u))} \subset \varphi_{(t,x)}, \quad \psi_{(t+u, \psi_{(t,x)}(t+u))} \subset \psi_{(t,x)}$$

for every $u \in R$ such that $u+t \in \Delta_{(t,x)}$ (for any $h_i: \Delta_i \rightarrow R$, $\Delta_i \subset R$, $i = 1, 2$ we put: $h_1 \subset h_2 \Leftrightarrow \Delta_1 \subset \Delta_2$ and $h_1(x) = h_2(x)$ for every $x \in \Delta_1$; $h_1 \leq h_2 \Leftrightarrow \Delta_1 = \Delta_2$ and $h_1(x) \leq h_2(x)$ for every $x \in \Delta_1$). Then the pair (Φ, Ψ) defines the generalized local pseudo-dynamical system (R^2, R_*, A, λ) , where $A = \{(u, (t, x)): u+t \in \Delta_{(t,x)}\}$ (it is obvious that $I_{(t,x)} = \{u: u+t \in \Delta_{(t,x)}\}$ in this case) and $\lambda(u, (t, x)) = \{t+u\} \times [\varphi_{(t,x)}(t+u), \psi_{(t,x)}(t+u)]$. We will denote by \mathcal{X} the class of the systems on the plane which can be obtained in this way.

Remark 2. This class is a generalization of the generalized local pseudo-dynamical systems obtained by differential equations $x' = f(t, x)$, where f is a continuous function from R^2 into R . In this case $\varphi_{(t,x)}$ and $\psi_{(t,x)}$ we interpret as the right-hand minimum solution and the right-hand maximum solution of this equation at the point (t, x) .

Let a generalized local pseudo-dynamical system (R^2, R_*, A, λ) be given. Let M be a non-empty subset of the plane. We put

$$\beta: R^2 \ni z \rightarrow \{K(z, \varepsilon)\}_{\varepsilon > 0} \in \mathcal{P}(\mathcal{P}(R^2)) \quad \text{and} \quad \Omega = \{K(M, \varepsilon)\}_{\varepsilon > 0}.$$

DEFINITION 3. The set M is said to be a *weak stable* set if and only if it is $(ZOdH, \Omega, \beta)$ -weak stable. Every $(ZOdH, \Omega, \beta)$ -weak Lyapunov function $V: W \rightarrow R_*$ will be called the *weak Lyapunov function*.

We note that M is weak stable if and only if for every $z \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda_t(w) \cap K(M, \varepsilon) \neq \emptyset$ for every $w \in K(z, \delta)$ and $t \in I_w$. This stability is a generalization of the classical Lyapunov stability in the generalized local pseudo-dynamical systems.

Now let us suppose that (R^2, R_*, A, λ) is a system of the class \mathcal{K} . Assume in the sequel that the set M has the form

$$M = \{(\tau, y) : \tau \in R, \alpha(\tau) \leq y \leq \beta(\tau)\},$$

where α and β are the continuous functions from R into R such that $\alpha(w) \leq \beta(w)$ for each $w \in R$. Then the condition of the weak stability of M has the form: for each $(s, x) \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\varphi_{(t,y)}(t+u), \psi_{(t,y)}(t+u)] \cap [\alpha(t+u) - \varepsilon, \beta(t+u) + \varepsilon] \neq \emptyset$ for every $(t, y) \in K((s, x), \delta)$ and $u \in I_{(t,y)}$. We have

$$[\varphi_{(s,x)}(s+u), \psi_{(s,x)}(s+u)] \cap [\alpha(s+u), \beta(s+u)] \neq \emptyset$$

for every $(s, x) \in M$ and $u \in I_{(s,x)}$, in particular.

Remark 3. The last condition means that M is a weak invariant set whenever it is weak stable.

LEMMA. Assume that M is a weak stable set. If for any $(s, x) \notin M$ we have $x < \alpha(s)$, then $\varphi_{(s,x)}(w) \leq \beta(w)$ for every $w \in \Delta_{(s,x)}$. Similarly, if for any $(s, x) \notin M$ we have $x > \beta(s)$, then $\psi_{(s,x)}(w) \geq \alpha(w)$ for every $w \in \Delta_{(s,x)}$.

Proof. Let $(s, x) \notin M$ be such that $x < \alpha(s)$. Let us assume, to the contrary, that $\varphi_{(s,x)}(w_0) > \beta(w_0)$ for some $w_0 \in \Delta_{(s,x)}$. From the continuity of the function $\varphi_{(s,x)}$ we have that there exists $w_1 \in \Delta_{(s,x)}$, $w_1 \leq w_0$ such that the point $A = (w_1, \varphi_{(s,x)}(w_1))$ is in the set M . By the definition of generalized local pseudo-dynamical system we obtain $[\varphi_A(w_0), \psi_A(w_0)] \cap [\alpha(w_0), \beta(w_0)] = \emptyset$.

This contradicts Remark 3. The argumentation in the other case is similar. The proof of the lemma is completed.

THEOREM 3. If the set M is weak stable then there exists $W \supset M$ and a weak Lyapunov function $V: W \rightarrow R_*$ for M .

Proof. We define

$$W = \{(s, x) : \exists r > 0 \forall u \in I_{(s,x)}, [\varphi_{(s,x)}(s+u), \psi_{(s,x)}(s+u)] \cap [\alpha(s+u) - r, \beta(s+u) + r] \neq \emptyset$$

and

$$V(s, x) = \inf\{r > 0 :$$

$$\forall u \in I_{(s,x)}, [\varphi_{(s,x)}(s+u), \psi_{(s,x)}(s+u)] \cap [\alpha(s+u) - r, \beta(s+u) + r] \neq \emptyset\}.$$

To prove Theorem 3 we must show that V fulfils the following conditions:

- (j) for each $(s, x) \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $(t, y) \in W$ and $V(t, y) \leq \varepsilon$ for every $(t, y) \in K((s, x), \delta)$,
- (jj) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $(t, y) \in W$ and $V(t, y) \leq \delta$ then $y \in [\alpha(t) - \varepsilon, \beta(t) + \varepsilon]$,

(jjj) for each $(s, x) \in W$ and $u \in I_{(s, x)}$ there exists $y \in R$ such that $(s+u, y) \in W$, $y \in [\varphi_{(s, x)}(s+u), \psi_{(s, x)}(s+u)]$ and $V(s+u, y) \leq V(s, x)$.

The definition of the weak stability of M implies immediately that the condition (j) is satisfied. Now let us show (jj). For an arbitrarily fixed $\varepsilon > 0$ we can take any $\delta \in (0, \varepsilon)$. Let $(t, y) \in W$ and $V(t, y) \leq \delta < \varepsilon$. Then there is $\varepsilon_0 \in [V(t, y), \varepsilon)$ such that

$$[\varphi_{(t, y)}(t+u), \psi_{(t, y)}(t+u)] \cap [\alpha(t+u) - \varepsilon_0, \beta(t+u) + \varepsilon_0] \neq \emptyset.$$

By putting $u = 0$ we have $y \in [\alpha(t) - \varepsilon_0, \beta(t) + \varepsilon_0]$, moreover $y \in [\alpha(t) - \varepsilon, \beta(t) + \varepsilon]$. The condition (jj) is proved. Now we shall show that (jjj) is satisfied. Note first that $(t, y) \in W$ and $V(t, y) = 0$ for every $(t, y) \in M$. It is a simple consequence of Remark 3. Let now $(s, x) \in W$ and $u \in I_{(s, x)}$ be given. Denote $r_0 = V(s, x)$. It is easy to see that

$$[\varphi_{(s, x)}(s+u), \psi_{(s, x)}(s+u)] \cap [\alpha(s+u) - r_0, \beta(s+u) + r_0] \neq \emptyset.$$

In the case $(s, x) \in M$ the proof is trivial. Let us suppose that $(s, x) \notin M$ and let for instance $\beta(s) < x$ (if $x < \alpha(s)$ then the argumentation is similar). From Lemma it follows that $\psi_{(s, x)}(s+u) \geq \alpha(s+u)$. If $\varphi_{(s, x)}(s+u) \leq \beta(s+u)$, then there exists $y \in [\varphi_{(s, x)}(s+u), \psi_{(s, x)}(s+u)] \cap [\alpha(s+u), \beta(s+u)]$. Then $(s+u, y) \in M$ and $(s+u, y) \in W$, $V(s+u, y) = 0$. In this case (jjj) is satisfied. Let now $\varphi_{(s, x)}(s+u) > \beta(s+u)$. We show that the point $y_0 = \varphi_{(s, x)}(s+u)$ is such that $(s+u, y_0) \in W$ and $V(s+u, y_0) \leq V(s, x)$. It suffices to prove that

$$[\varphi_{(s+u, y_0)}(s+u+w), \psi_{(s+u, y_0)}(s+u+w)] \cap [\alpha(s+u+w) - r_0, \beta(s+u+w) + r_0] \neq \emptyset$$

for every $w \in I_{(s+u, y_0)}$. Suppose, to the contrary, that the considered set is empty. From Lemma it follows that

$$\beta(s+u+w_0) + r_0 < \varphi_{(s+u, y_0)}(s+u+w_0).$$

By the definition of the class \mathcal{K} we obtain $\varphi_{(s, x)}(s+u+w_0) = \varphi_{(s+u, y_0)}(s+u+w_0) > \beta(s+u+w_0) + r_0$. On the other hand, since $r_0 = V(s, x)$ we have

$$[\varphi_{(s, x)}(s+u+w_0), \psi_{(s, x)}(s+u+w_0)] \cap [\alpha(s+u+w_0) - r_0, \beta(s+u+w_0) + r_0] \neq \emptyset.$$

which is a contradiction. Therefore $(s+u, y_0) \in W$, $V(s+u, y_0) \leq V(s, x)$. That gives the condition (jjj). The proof of Theorem 3 is completed.

5. As some application we have following

THEOREM 4. Let (R^2, R_*, A, λ) be a generalized local pseudo-dynamical system generated by the differential equation $x' = f(t, x)$, where f is a continuous function from R^2 into R (see Remark 2). Let us suppose that the set $M \subset R^2$ has the form

$$M = \{(t, x) \in R^2: t \in R, x \in [\alpha(t), \beta(t)]\}$$

for any continuous function $\alpha, \beta: R \rightarrow R$, $\alpha \leq \beta$. Then the set M is weak stable if and only if there exists a set $W \supset M$ and a function $V: W \rightarrow R_*$ fulfilling the following conditions:

(k) f vanishes on the set M and is continuous at every point of M ,

(kk) for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $(t, y) \in W$ and $V(t, y) \leq \delta$ then $y \in [\alpha(t) - \varepsilon, \beta(t) + \varepsilon]$,

(kkk) for each $(s, x) \in W$ and $u \in I_{(s, x)}$ there exists $y \in R$ such that $(s+u, y) \in W$, $y \in [\varphi_{(s, x)}(s+u), \psi_{(s, x)}(s+u)]$ and $V(s+u, y) \leq V(s, x)$.

This theorem is a simple consequence of the proof of Theorem 3 and Theorem 2.

6. Remark to [11]. The following assumption should be added to the content of the paper [11]: there exists $\omega \in \Omega$ containing an invariant non-empty subset. I would like to express my thanks to P. Tworzewski for his remark.

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