ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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Some Remarks on Processes III

Jerzy OMBACH and Andrzej TRZEPIZUR

We give theorem which says that for a limit pseudoprocess v of the pseudoprocess generated by an ordinary differential equation $x' = f(t, x)$ there is a differential equation $x' = g(t, x)$ generating v. This is an inverse problem of the problem discussed in [5]. We shall use the notation and terminology from [3], [4], [5].

Let $f\colon R\times R^n \to R^n$ be a continuous mapping. We consider a differential equation

$$x' = f(t, x)\,. \tag{1}$$

We assume that for every $(t_0, x_0)\in R\times R^n$ there is the solution $\varphi(t_0, x_0, t)$ of equation (1) with the initial condition $x(t_0) = x_0$, defined for $t\in[t_0, \infty)$. We define the function $u\colon R\times R^n\times R_* \to R^n$ by $u^t(x, \tau) = \varphi(t, x, t+\tau)$ (we will use the notation $u^t(x, \tau)$ instead $u(t, x, \tau)$).

In the case if the problem $x' = f(t, x)$, $x(t_0) = x_0$ has exactly one solution defined for $t\in[t_0, \infty)$, for all $(t_0, x_0)\in R\times R^n$ we call u a *pseudoprocess* generated by the equation (1) (see [3], [4], [5]). For $T\in R$ we define the function $u_T\colon R\times R^n\times R_*\ni(t, x, \tau) \to u^{t+T}(x, \tau)\in R$. If u is the pseudoprocess, u_T is said to be a *T-translation* of u.

THEOREM. *Suppose that the mapping f is bounded and uniformly continuous. Let $v\colon R\times R^n\times R_*\to R^n$ be a function such that for some sequence $\{T_n\}$, $T_n\in R$ we have $u_{T_n}\to v$, as $n\to\infty$ at every point $(t, x, \tau)\in R\times R^n\times R_*$. Then there exists a continuous function $g\colon R\times R^n \to R^n$ and a solution $\psi(t_0, x_0, t)$, $t\in[t_0, \infty)$ of the initial problem*

$$\begin{aligned} x' &= g(t, x)\\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

such that $v^t(x, \tau) = \psi(t, x, t+\tau)$ for $(t, x, \tau)\in R\times R^n\times R_*$ and $f(t+T_n, x)\to g(t, x)$ as $n\to\infty$, uniformly on compact sets in $R\times R^n$.

If we assume additionally that f fulfils the local Lipschitz condition with respect to x, then u and v are the pseudoprocesses obtained from the differential equations (1) *and* (2), *respectively, and in the case $T_n\to\infty$, v is the limit pseudoprocess of u (see* [5], [1], [2]).

First we shall prove the following:

Lemma. *Let $p_n\colon R \to R$ for $n = 1, 2, \ldots$, be a sequence of the differentiable functions such that for some function $p\colon R \to R$*

$$p_n(s) \to p(s)\,, \qquad \textit{as } n \to \infty$$

at each point $s \in R$. If the sequence $\{p'_n\}_{n=1,2,\ldots}$ is a commonly bounded and uniformly equicontinuous family of the functions, then the derivative p' exists, moreover $p'_n \to p'$ uniformly on compact sets in R.

Proof. The sequence $\{p'_n\}$ fulfils the assumptions of the Arzeli Theorem, therefore there is a subsequence $\{p_{n_k}\}$ and a function q such that $p'_{n_k} \to q$ uniformly on compact sets in R. Hence, by the known theorem, the derivative p' exists and $p' = q$. By the same argument we can show that for every sequence $\{l_k\}$ there exists a subsequence $\{l_{n_k}\}$ such that $p'_{l_{n_k}} \to q$ uniformly on compact sets in R. The Lemma is proved.

Now we prove the Theorem. Because $\varphi(t_0, x_0, t)$, $t \geqslant t_0$ is the solution of (1) with the initial condition $x(t_0) = x_0$, then we have

$$\frac{\delta u^{t_0}(x_0, \tau)}{\partial \tau} = f(t_0+\tau, \varphi(t_0, x_0, t_0+\tau)\,.$$

We put

$$p_n(\tau) = u^{t_0+T_n}(x_0, \tau)\,.$$

Since f is a bounded function, the function $\varphi(t_0, x_0, t)$ fulfils the Lipschitz condition with the same constant, with respect to t. Thus by the uniform continuity of f and previous property of the function $\varphi(t_0, x_0, t)$ we obtain that the sequence $\{p'_n\}$ is uniformly equicontinuous family. Because f is bounded, the family $\{p'_n\}$ is commonly bounded. Now, using Lemma we see that there exists the derivative $\dfrac{\partial v^{t_0}(x_0, \tau)}{\partial \tau}$ and

$$\frac{\partial u^{t_0+T_n}(x_0, \tau)}{\partial \tau} \to \frac{\partial v^{t_0}(x_0, \tau)}{\partial \tau}$$

uniformly on compact sets in R. We define

$$g(t, x) = \left.\frac{\partial v^{t}(x, \tau)}{\delta \tau}\right|_{\tau=0}\,.$$

It is obvious that $f(t+T_n, y) \to g(t, y)$, as $n \to \infty$, for each $(t, y) \in R \times R^n$. Moreover, by the Arzeli Theorem this convergence is uniform on compact sets in $R \times R^n$. Then g is a continuous and g fulfils the local Lipschitz condition while f its fulfils. By inequality

$$\begin{aligned}
&|f(t_0+T_n, u^{t_0+T_n}(x_0, \tau)) - g(t_0, v^{t_0}(x_0, \tau))| \\
&\qquad \leqslant |f(t_0+T_n, u^{t_0+T_n}(x_0, \tau)) - f(t_0+T_n, v^{t_0}(x_0, \tau))| + \\
&\qquad\qquad + |f(t_0+T_n, v^{t_0}(x_0, \tau)) - g(t_0, v^{t_0}(x_0, \tau))|
\end{aligned}$$

and by the uniform continuity of the function f we can pass to the limit in the equality

$$\frac{\partial u^{t_0+T_n}(x_0, \tau)}{\partial \tau} = f(t_0+T_n, u^{t_0+T_n}(x_0, \tau))$$