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On Period Three and Chaos

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1. Let $f\colon [0,1] \to [0,1]$ be a continuous function. For a positive integer n define the n-th iteration f^n of f by $f^{n+1}(x) = f(f^n(x)), f^0(x) = x$. Thus we obtain a semi-dynamical system on the interval $[0,1]$. Such systems are often used as the mathematical models of biological processes, see for example [5], [4], [7]. In almost all such situations the following conditions are assumed.

(1) $f(0) = f(1) = 0$

(2) *there is a unique* $c \in (0,1)$ *such that* f *increases in* $[0,c]$ *and decreases in* $[c,1]$.

In any cases the function f depends on one or more parameters and then the following phenomenon has been observed ([3], [5]). For some, we call them "small", values of these parameters there is a unique non-zero fixed point of f which is stable (the necessary definitions are given below). As parameters change, this point ceases to be stable and bifurcates to a periodic orbit with period 2, and in turn to a periodic orbits with periods 4, 8, 16, ... There orbits are stable in some situations. For some, we call them "large", values of parameters it is hard to decide whether the stable orbits exist ([3], [5], [8]). Such situation is often called *chaos*. There are many definitions of chaos, but we will not discuss them here. For biologists there is *chaos*, if there are infinitely many unstable periodic orbits (see [5]) and we will understand by chaos just the same.

T. V. Li and J. A. Yorke have been proved in [3] that the existence of a periodic orbit with period three implies chaos. In other papers (see [2]) it was proved that in this case there is a measure invariant under f. It would be very useful to obtain some statistical properties of the system in the case mentioned above, (see [1]). It seems that for the reasons mentioned above, notions of chaos and "period three" are too strongly connected, see [5]. Two examples that are given in this note indicate that this need not be so.

2. First we recall some definitions and give simple properties needed in the examples. Let $I = [0,1]$. An invariant set $A \subset I$ (i.e. $f(A) \subset A$) is said to be: *stable*, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for x from the δ-neighbourhood of A, $f^n(x)$ belongs to the ε-neighbourhood of A for $n = 1, 2, 3, \ldots$, *asymptotically stable*, if it is stable and for x close to A, $f^n(x) \to A$ as $n \to \infty$, *globally asymptotically stable* in $I_1 \subset I$, if additionally a domain of attraction of A i.e. the set $\{x \in I_1 : f^n(x) \to A, \text{ as } n \to \infty\}$ coincides with I_1. A point $p \in I$ is said to be *periodic with period n* if $f^n(p) = p$ and $f^k(p) \neq p$ for $1 \leqslant k < n$. In this case the set $o(p) = \{p, f(p), \ldots, f^{n-1}(p)\}$ is said to be a *periodic orbit* of the point p. Let $J \subset R$ be a closed interval and $g: J \to J$ be a continuous function. We say that g is *topologically equivalent* to $f: I \to I$ if there is a homeomorphism $h: I \to J$ such that $h \circ f = g \circ h$. If h has a form $h(x) = ax + b$, $a, b \in R$ we say that f and g are *linearly equivalent.*

The following lemmas are known and not hard to prove.

Lemma 1. *The periodic point p with period n gives the stable (asymptotically stable) periodic orbit with respect to f, if and only if p is the stable (asymptotically stable) fixed point with respect to the function* f^n.

Lemma 2. *Let* $p \in (0, 1)$ *be a fixed point of f, and f be of the class* C^1 *in p. Then: for* $|f'(p)| < 1$, *p is asymptotically stable, for* $|f'(p)| > 1$, *p is unstable.*

Lemma 3. *Let* $p \in (0, 1)$ *be a periodic point of f with period n, and f be of the class* C^1 *in the points of the orbit o(p). Then o(p) is asymptotically stable if* $|\lambda(p)| < 1$, *and unstable if* $|\lambda(p)| > 1$, *where* $\lambda(p) = (f^n)'(p) = f'(f^{n-1}(p)) \times \ldots \times f'(p)$.

3. In the sequel we will use the conditions (1), (2) and the following

(3) *there is a unique non-zero fixed point of f, this point* $p > c$.

Proposition. *We assume that the function f satisfies the conditions* (1), (2), (3) *and that f is a contraction on the interval* $[c, f(c)]$. *Then the point p is globally asymptotically stable in* $(0, 1)$.

Proof. Since $f([c, f(c)]) \subset [c, f(c)]$ we have the asymptotic stability of p with the domain of attraction containing $[c, f(c)]$. Let $0 < x < c$, then by (2) $f(x) < f(c)$. If for all n $f^n(x) < c$, then $x < f(x) < f^2(x) < \ldots$, and there is a point $q \leqslant c$ such that $f(q) = q$, which contradicts to (3). Hence there is an integer n such that $f^n(x) \in [c, f(c)]$ which shows that the domain of attraction of p contains $(0, c)$. Similarly, if $f(c) < x < 1$, then $c < x$ and $f(x) < f(c)$, whence again there is an integer n such that $f^n(x) \in [c, f(c)]$ and the proof is completed.

The Proposition is applied in Example 1 and in the papers [4] and [7]. Note that the function $f: I \to I$ defined as $f(x) = 3x(1-x)$ satisfies (1), (2), (3) and p is globally asymptotically stable in $(0, 1)$ and yet f is not a contraction on $[c, f(c)]$ $[0.5, 0.75]$ as $f'(0.75) = -1.5$. There is a criterion for globally asymptotic stability better than Proposition, see [6].

Lemma 4. *Let the conditions* (1), (2), (3) *be satisfied. If the point p is unstable, then there are periodic points with period two.*

Proof. Notice that there is a number $a<p$ such that $f^2(c)<a$ or there is a number $b>p$ such that $f^2(b)>b$: otherwise for $x \in (c,p)$ we would have: $p<f(x)$, $x \leqslant f^2(x)<p$, $p<f^3(x) \leqslant f(x)$, etc., contrarily to the unstability of p. For a_1 close to 0 and b_1 close to 1 we have $f^2(a_1)>a_1$ and $f^2(b_1)<b_1$, which by continuity of f^2 implies the assertion of Lemma 4.

We will assume the condition:

(4) *f is linear of the intervals* $[0,c]$ *and* $[c,1]$.

LEMMA 5. *If the conditions* (1)–(4) *are satisfied and the point p is unstable, then for any natural n there is a periodic orbit with period 2^n. Every periodic orbit is unstable.*

Proof. We apply Lemma 4 and obtain the periodic orbit $o(p_1)$ with period two. By condition (4) and the unstability of p we have $|\lambda(p_1)|>1$, and by Lemma 3 $o(p_1)$ is unstable. Let us consider the interval $J=[p,p_r]$, where $p<p_r$ and $f^2(p_r)=p$. Obviously either p_1 or $f(p_1)$ belongs to J. Let $g(x)=f^2(x)$ for $x \in I$, and let g_1 is the restriction of g to J. We assume for a moment that $g(J) \subset J$. The function g_1 is linearly equivalent to a function $g_2\colon I \to I$ satisfying the conditions (1)–(4) and such that the fixed point of g_2 is unstable. By Lemma 4 g_2, g_1 and g have a periodic orbit with period two which is unstable. This orbit forms a periodic orbit with respect to g with period four. If $g(J) \not\subset J$, then by continuity of the function g we obtain a point a such that $p<a<p_1$, and $g(a)=p_1$. Hence $g^3(a)=p<a$. Since p is unstable there is a point b such that $p<b<a$ and $g^3(b)>b$. So there is a point with period three with respect to g. By the result of [3] g has periodic points with every periods, whence f has periodic points with periods $2n$ and therefore with periods 2^n, $n=2,3,4,\ldots$ The induction completes the proof of the first statement. The second one follows from Lemma 3 and the condition (4) (if the point c is a periodic point we may use in Lemma 3 one-sided derivatives).

LEMMA 6. *If the conditions* (1), (2), (3) *are satisfied and $f^3(c) \leqslant c$, then there is a periodic point with period three.*

The proof is a simple consequence of the continuity of the function f^3.

The condition $f^3(c) \leqslant c$ is not necessary for a "period three". For the function $f\colon I \to I$ defined as $f(x)=3.83x(1-x)$ considered in [8] there is a periodic point with period three, but $c=0.5$, $f^3(c)=0.503896$.

LEMMA 7. *If f satisfies the conditions* (1)–(4), *$f^3(c) \leqslant c$ if and only if there is a periodic point with period three.*

The proof is a consequence of the piecewise linearity of the function f^3.

4. Example 1. A. Łomnicki in [4] has used the following recurrence relation to describe the development growth of the animal population:

$$x_{n+1}=f(x_n), \quad n=0,1,2,\ldots$$

where x_n means the number of individuals in n-th generation and the function f is given as

$$f(x) = \begin{cases} h(a-m)x & \text{for } 0 \leqslant x \leqslant \dfrac{V}{a} \\ h(V-mx) & \text{for } \dfrac{V}{a} \leqslant x \leqslant \dfrac{V}{m} \end{cases}$$

where h, a, m, V are positive constants and $m<a<V$. For simplicity we put (see [7]): $M = h(a-m)$, $C = \dfrac{V}{a}$, $p = \dfrac{m}{a}$. The parameters M, p, C can be interpreted respectively as: the maximum offsprings produced by an individual, the ability to obtain progeny, and the maximum number of individuals such that all individuals can take food without limitation.

The system is well-defined provided

$$Mp \leqslant 1 . \tag{5}$$

The above system can be identified with the linearly conjugate system defined on I. We see that the conditions (1), (2), (4) are satisfied. The condition (3) is satisfied if

$$M > 1 . \tag{6}$$

The globally asymptotic stability of the fixed point p is equivalent to the condition

$$\frac{Mp}{1-p} < 1 . \tag{7}$$

We now assume the conditions (5), (6) and

$$\frac{Mp}{1-p} > 1 . \tag{8}$$

By Lemma 5 there is chaos. On the other hand we may use Lemma 7 which implies that there is a periodic point with period three iff

$$\frac{M^2}{1-p}(1-Mp) \leqslant 1 \tag{9}$$

There are many values of parameters M and p satisfying (5), (6), (8) but not (9). For example $M = 4$, $p = \frac{2}{9}$. Hence it may be chaos without "period three". It is necessary to remark that under assumptions (5), (6), (8) there is an invariant measure under f (see [3]). Hence nice statistical property can be obtained.

Example 2. Let $0<a<0.5$ and let $f\colon I \to I$ be continuous and linear on intervals: $\left[0, \frac{a}{2}\right]$, $\left[\frac{a}{2}, a\right]$, $[a, 1-a]$, $[1-a, 1]$ satisfying $f(0) = f(1) = 0$, $f\left(\frac{a}{2}\right) = 1$, $f(a) = 1-a$, $f(1-a) = 3a$.

Clearly f satisfies (1), (2), (3). Since $f^3\left(\frac{a}{2}\right) = 0 < \frac{a}{2}$, by Lemma 6 there is a periodic point with period three and hence chaos. But it is rather “a local chaos”. We see that $p \in (a, 1-a)$ and the interval $(a, 1-a)$ is an invariant set. The point p is asymptotically stable with the domain of attraction containing $(a, 1-a)$. Thus for a close to 0 the domain of attraction is close to I.

Let us remark that in both examples the function f is not differentiable but this fact is inessential. Recently I have been finding that the effect given by Example 1 holds also for the functions $f_t(x) = tx(1-x)$ for $t \in (3.570\ldots, 3.679\ldots)$. On other hand Professor A. Lasota called my attention to the function $f(x) = 4x^2(1-x^2)$ giving the effect similar to this in Example 2.

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