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Strong and Weak Non-linear Parabolic Differential-functional Inequalities with Functional Boundary Conditions

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1. Definitions and assumptions. Let $G\subset \mathbf{R}^n$ be an open and bounded set and let $D = [0,T)\times G$ for some $T>0$. Denote $\Sigma_0 = \{0\}\times \overline{G}$, $\Gamma = (0,T)\times \partial G$. Assume that $\Gamma = \Sigma \cup \Sigma^*$ (one of these sets may be empty), where Σ denotes the set of all points $(t,x)\in\Gamma$ which can be attained from D by some segment $l(t,x)$, orthogonal to the t-axis. Fix also an open neighborhood $\tilde{D}\subset \mathbf{R}^{n+1}$ of the set $\overline{D}$ and define the set $D_0 = \tilde{D}\cap [(-\infty, T)\times \mathbf{R}^n]$.

We introduce, following [1]

DEFINITION 1. The class $Z(D)$ *of admissible functions* is the set of functions $w\colon D_0\to\mathbf{R}$ such that:

(i) w is continuous in $[0,T)\times\overline{G}$;
(ii) the partial derivative w_t exists in $\operatorname{int} D$;
(iii) for every $t\in(0,T)$, the function $G\ni x\to w(t,x)\in\mathbf{R}$ is of the class C^2 in G.

Let two mappings be given

$$f\colon\ D\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^{n^2}\times Z(D)\to\mathbf{R},$$
$$\varphi\colon\ \Sigma\times\mathbf{R}\times\mathbf{R}\times Z(D)\to\mathbf{R}.$$

DEFINITION 2. An *interior operator* P and *boundary operator* R are defined by the formulas

$$Pw(t,x) = w_t(t,x)-f(t,x,w(t,x),w_x(t,x),w_{xx}(t,x),w),\qquad w\in Z(D),\quad (t,x)\in\operatorname{Int}D,$$

$$Rw(t,x) = w(t,x)-\varphi\Big(t,x,w(t,x),\frac{\partial w}{\partial l}(t,x),w\Big),\qquad w\in Z(D),\quad (t,x)\in\Sigma,$$

$$\text{where } \frac{\partial w}{\partial l}(t,x) = \liminf_{\tau\to 0+}\frac{w\big(t,x+\tau\operatorname{vers}l(t,x)\big)-w(t,x)}{\tau}.$$

Remark. The operator P is defined analogously to the one considered in [1] while in the formula for the boundary operator R the functional argument is added. The boundary conditions of this kind occur in some technical problems.

Finally, we formulate some monotonicity properties for the functions f and φ.

Assumptions (M).

(a) $\forall w,\ \tilde{w} \in Z(D)$ and $t^* \in [0, T)$ the following implications hold true:

(I) $\forall x \in G,\ a \in \boldsymbol{R},\ b \in \boldsymbol{R}^n,\ c \in \boldsymbol{R}^{n^2}$:

if $w \leqslant \tilde{w}$ in the set $[0, t^*) \times G$ then $f(t^*, x, a, b, c, w) \leqslant f(t^*, x, a, b, c, \tilde{w})$.

(II) $\forall x \in \partial G$, such that $(t^*, x) \in \Sigma$, $\forall a, b \in \boldsymbol{R}$:

if $w \leqslant \tilde{w}$ in the set $([0, t^*) \times \boldsymbol{R}^n) \cap \Sigma$ then $\varphi(t^*, x, a, b, w) \leqslant \varphi(t^*, x, a, b, \tilde{w})$.

(b) (I′) $\forall (t, x) \in D,\ a \in \boldsymbol{R},\ b \in \boldsymbol{R}^n,\ w \in Z(D)$:

if $c, \tilde{c} \in \boldsymbol{R}^{n^2}$, $c \leqslant \tilde{c}$, i.e. $\sum_{i,j} (c_{ij} - \tilde{c}_{ij}) \lambda_i \lambda_j \leqslant 0$ then $f(t, x, a, b, c, w) \leqslant f(t, x, a, b, \tilde{c}, w)$.

(II′) $\forall (t, x) \in \Sigma,\ a \in \boldsymbol{R},\ w \in Z(D)$:

if $b \leqslant \tilde{b}$ in $\boldsymbol{R}$ then $\varphi(t, x, a, b, w) \leqslant \varphi(t, x, a, \tilde{b}, w)$.

2. Strong and weak inequalities.

Two theorems presented in this section generalize, in the case of bounded cylindrical domain, the respective results by Szarski [3]. The method of our proofs is inspired by [3].

THEOREM 1 (strong inequalities). *Let* $u, v \in Z(D)$. *Assume that*

1° f *and* φ *satisfy* Assumptions (M);

2° $Pu < Pv$ *in* $\operatorname{int} D$;

3° $u < v$ *in* $\Sigma_0 \cup \Sigma^*$;

4° $Ru < Rv$ *in* Σ.

Then $u < v$ *in* D.

Proof. Denote $w = u - v$ and $\Delta = \{t^* \in (0, T)\colon w < 0 \text{ in } [0, t^*) \times G\}$. Observe that $\Delta \neq \emptyset$. Indeed, the continuous function w is negative in the compact set Σ_0 (by 3°) and so there exists $t^* > 0$ such that $t^* \in \Delta$.

Let $\tilde{t} = \sup \Delta$. The desired statement is obviously equivalent to the equality $\tilde{t} = T$. Suppose that $\tilde{t} < T$ and put $S_{\tilde{t}} = \{\tilde{t}\} \times \bar{G}$. By the definition of $\tilde{t}$ and by the continuity of w in $[0, T) \times \bar{G}$ we obtain the inequality $w \leqslant 0$ in the set $S_{\tilde{t}}$.

Assume that there exists $\tilde{x} \in G$ such that $w(\tilde{t}, \tilde{x}) = 0$. Therefore the function (of the class C^2 in G) $G \ni x \to w(t, x) \in \boldsymbol{R}$ attains its maximum in the point $\tilde{x}$. Hence we have

$$(1) \qquad \begin{gathered} w_x(\tilde{t}, \tilde{x}) = 0, \qquad w_{xx}(\tilde{t}, \tilde{x}) \leqslant 0 \\ \text{i.e.} \\ u_x(\tilde{t}, \tilde{x}) = v_x(\tilde{t}, \tilde{x}), \quad u_{xx}(\tilde{t}, \tilde{x}) \leqslant v_{xx}(\tilde{t}, \tilde{x}). \end{gathered}$$

By 1°, 2° and (1), we get successively

$$
\begin{aligned}
(2)\qquad & w_t(\tilde t,\tilde x)<f(\tilde t,\tilde x,u(\tilde t,\tilde x),u_x(\tilde t,\tilde x),u_{xx}(\tilde t,\tilde x),u)-\\
&\quad -f(\tilde t,\tilde x,v(\tilde t,\tilde x),v_x(\tilde t,\tilde x),v_{xx}(\tilde t,\tilde x),v)\leqslant\\
&\quad \leqslant f(\tilde t,\tilde x,v(\tilde t,\tilde x),v_x(\tilde t,\tilde x),v_{xx}(\tilde t,\tilde x),u)-\\
&\quad -f(\tilde t,\tilde x,v(\tilde t,\tilde x),v_x(\tilde t,\tilde x),v_{xx}(\tilde t,\tilde x),v)\leqslant 0\,.
\end{aligned}
$$

On the other hand, the function $(0,\tilde t]\ni t\to w(t,\tilde x)\in \boldsymbol{R}$ attains its maximum in the point $\tilde t$. Hence $w_t(\tilde t,\tilde x)\geqslant 0$, what contradicts (2).

Thus we have the inequality $w<0$ in the set $\{\tilde t\}\times G$.

Assume in turn that there exists $\tilde x\in\partial G$ such that $w(\tilde t,\tilde x)=0$. By 3° it must be $(\tilde t,\tilde x)\in\Sigma$ and by the definition of lower derivative in the direction $l(\tilde t,\tilde x)$ we obtain the inequality

$$
(3)\qquad \frac{\partial w}{\partial l}(\tilde t,\tilde x)\leqslant 0.
$$

By 1°, 4° and (3), we get successively

$$
\begin{aligned}
0=w(\tilde t,\tilde x)&<\varphi\Big(\tilde t,\tilde x,u(\tilde t,\tilde x),\frac{\partial u}{\partial l}(\tilde t,\tilde x),u\Big)-\varphi\Big(\tilde t,\tilde x,v(\tilde t,\tilde x),\frac{\partial v}{\partial l}(\tilde t,\tilde x),v\Big)\leqslant\\
&\leqslant\varphi\Big(\tilde t,\tilde x,u(\tilde t,\tilde x),\frac{\partial u}{\partial l}(\tilde t,\tilde x),u\Big)-\varphi\Big(\tilde t,\tilde x,u(\tilde t,\tilde x),\frac{\partial u}{\partial l}(\tilde t,\tilde x),v\Big)\leqslant 0\,.
\end{aligned}
$$

This contradiction proves the inequality $w<0$ in $S_{\tilde t}$.

By the argument of compactness of $S_{\tilde t}$ and the continuity of w we can find $t_1>\tilde t$ such that $w<0$ in the set $[0,t_1)\times G$. This contradicts the definition of $\tilde t$ and completes the proof.

THEOREM 2 (weak inequalities). *Let* $u,v\in Z(D)$. *Assume that*

1° f *and* φ *satisfy* Assumptions (M);

2° $\forall (t,x)\in D$, $a,\tilde a\in\boldsymbol{R}$, $\tilde a\leqslant a$, $b\in\boldsymbol{R}^n$, $c\in\boldsymbol{R}^{n^2}$, $w,\tilde w\in Z(D)$ *we have*

$$
f(t,x,a,b,c,w)-f(t,x,\tilde a,b,c,\tilde w)\leqslant\sigma\big(t,\max(a-\tilde a,\|w-\tilde w\|_{D_t})\big),
$$

where $D_t=[0,t)\times G$ *and* $\sigma\colon \{t\geqslant 0\}\times\{y\geqslant 0\}\to\boldsymbol{R}_+\cup\{0\}$ *is continuous such that* $y(t)=0$ *is the unique solution of the differential equation* $\dfrac{dy}{dt}=\sigma(t,y)$, *satisfying the initial condition* $y(0)=0$.

3° $\forall (t,x)\in\Sigma$, $\exists K=K(t,x)$, $0<K<1$ *such that* $\forall a,\tilde a\in\boldsymbol{R}$, $\tilde a\leqslant a$, $b\in\boldsymbol{R}$, $w,\tilde w\in Z(D)$ *we have* $\varphi(t,x,a,b,w)-\varphi(t,x,\tilde a,b,\tilde w)\leqslant K\max(a-\tilde a,\|w-w\|_{\bar D_t\cap\Sigma})$;

4° $Pu\leqslant Pv$ *in* $\operatorname{int}D$;

5° $u\leqslant v$ *in* $\Sigma_0\cup\Sigma^*$;

6° $Ru\leqslant Rv$ *in* Σ.

Then $u\leqslant v$ *in* D.

Proof. Fix $0<T^*<T$. It is well known (cf [2], [3]) that there exists $\varepsilon_0>0$ such that for every $\varepsilon<\varepsilon_0$ we can find a differentiable function $y_\varepsilon:[0, T^*) \to \boldsymbol{R}$ with the following properties:

$$\begin{gathered}\frac{dy_\varepsilon}{dt} = \sigma(t, y_\varepsilon)+\varepsilon\,, \\ \lim_{\varepsilon\to 0} y_\varepsilon(t) = 0\,, \qquad t \in [0, T^*)\,, \\ y_\varepsilon(t)\geqslant\varepsilon\,, \qquad t\in[0, T^*)\,.\end{gathered} \tag{4}$$

Put $\tilde{v}(t,x) = v(t,x)+y_\varepsilon(t)$ for $(t,x)\in D_0 \cap ([0,T^*)\times \boldsymbol{R}^n)$ and $\varepsilon<\varepsilon_0$. Obviously $\tilde{v}\in Z(D_{T^*})$ and

$$\begin{gathered}\tilde{v}_x = v_x\,, \qquad \tilde{v}_{xx} = v_{xx} \qquad \text{in } \operatorname{int} D_{T^*}\,, \\ \frac{\partial \tilde{v}}{\partial l}(t,x) = \frac{\partial v}{\partial l}(t,x) \qquad \text{in } (t,x)\in \bar{D}_{T^*}\cap\Sigma\,.\end{gathered} \tag{5}$$

By the assumptions for the function f, the operator P and (4), (5) we get

$$\begin{aligned} P\tilde{v}(t,x) &= \tilde{v}_t(t,x)-f\big(t,x,\tilde{v}(t,x),\tilde{v}_x(t,x),\tilde{v}_{xx}(t,x),\tilde{v}\big) \\ &= v_t(t,x)+\frac{dy_\varepsilon}{dt}(t)-f\big(t,x,v(t,x)+y_\varepsilon(t),v_x(t,x),v_{xx}(t,x),v+y_\varepsilon\big) \\ &= v_t(t,x)+\sigma\big(t,y_\varepsilon(t)\big)+\varepsilon-f\big(t,x,v(t,x)+y_\varepsilon(t),v_x(t,x),v_{xx}(t,x),v+y_\varepsilon\big) \\ &\qquad \geqslant v_t(t,x)-f\big(t,x,v(t,x),v_x(t,x),v_{xx}(t,x),v\big)+\varepsilon>Pv(t,x)\geqslant Pu(t,x)\end{aligned}$$

or every $(t,x)\in\operatorname{int} D_{T^*}$.

In turn, by the assumptions for φ, R and (4), (5) we obtain

$$\begin{aligned} R\tilde{v}(t,x) &= \tilde{v}(t,x)-\Big(t,x,\tilde{v}(t,x),\frac{\partial\tilde{v}}{\partial l}(t,x),\tilde{v}\Big) \\ &= v(t,x)+y_\varepsilon(t)-\varphi\Big(t,x,v(t,x)+y_\varepsilon(t),\frac{\partial v}{\partial l}(t,x),v+y_\varepsilon\Big) \\ &\geqslant v(t,x)+y_\varepsilon(t)-\varphi\Big(t,x,v(t,x),\frac{\partial v}{\partial l}(t,x),v\Big)-Ky_\varepsilon(t) \\ &\geqslant Rv(t,x)+(1-K)\varepsilon>Rv(t,x)\geqslant Ru(t,x)\end{aligned}$$

for every $(t,x)\in\Sigma\cap\bar{D}_{T^*}$.

Therefore, all the assumptions of Theorem 1 are satisfied in the set D_{T^*}. Hence

$$u<v+y_\varepsilon \qquad \text{in } D_{T^*}$$

and by passing with ε to zero and using (4) we obtain the inequality $u\leqslant v$ in D_{T^*}. Since $0<T^*<T$ was fixed arbitrary, it follows that $u\leqslant v$ in D.

As an immediate consequence of Theorem 2 we obtain the following uniqueness criterion

THEOREM 3. *Let* $u, v \in Z(D)$. *Under the assumptions* 1,° 2°, 3° *of* Theorem 2 *suppose that*

(4) $Pu = Pv$ *in* $\operatorname{int} D$;

(5) $u = v$ *in* $\Sigma_0 \cup \Sigma^*$;

(6) $Ru = Rv$ *in* Σ.

Then $u = v$ *in* D.

References

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