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Almost Product and Almost Complex Structures Generated by Polynomial Structures

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It is known that a polynomial structure (in the sense given for example in [1]) defines an almost product structure. Moreover, if its characteristic polynomial P has only simple roots, then such a polynomial structure defines a (1,1) tensor field J satisfying the polynomial equation $J^3+J = 0$. J is an almost complex structure provided the polynomial P has no real root (see [1]).

The purpose of this note is to remark that these assertions also hold good in the case when coefficients of the characteristic polynomial are not necessarily constants but functions of class C^∞.

All the objects considered in this note are assumed to be smooth (C^∞) provided this is not to be shown.

Let M be a connected manifold. If U is an open subset in M then $C^\infty(U, \boldsymbol{R})$ ($C^\infty(U, \boldsymbol{C})$) denote the ring of all real (complex) valued functions on U. By $\boldsymbol{R}[z]$, $\boldsymbol{C}[z]$, $C^\infty(U, \boldsymbol{R})[z]$, $C^\infty(U, \boldsymbol{C})[z]$ we mean the polynomial rings over $\boldsymbol{R}$, $\boldsymbol{C}$, $C^\infty(U, \boldsymbol{R})$ and $C^\infty(U, \boldsymbol{C})$ respectively.

If

$$P(z) = a_0z^n+a_1z^{n-1}+\ldots+a_n \in C^\infty\big(M, \boldsymbol{R})[z](C^\infty(M, \boldsymbol{C})[z]\big),$$

then for every $x \in M$ we have the polynomial

$$P_x(z) \in \boldsymbol{R}[z](\boldsymbol{C}[z]), \quad P_x(z) = a_0(x)z^n+a_1(x)z^{n-1}+\ldots+a_n(x).$$

Let U be an open subset in M. $P(z)|_U$ will denote the polynomial belonging to $C^\infty(U, \boldsymbol{R})[z]\big(C^\infty(U, \boldsymbol{C})[z]\big)$ defined by

$$P(z)|_U = a_0|_Uz^n+a_1|_Uz^{n-1}+\ldots+a_n|_U.$$

If $P(z) \in C^\infty(M, \boldsymbol{C})[z]$ and $\xi \in C^\infty(U, \boldsymbol{C})$ be such that for every $x \in U$, $P_x(\xi(x)) = 0$, then ξ will be called a root function of the polynomial $P(z)$.

We shall consider a (1.1) tensor field f on M satisfying the polynomial equation

$$P(f) = 0 \tag{1}$$

i.e. $P_x(f_x) = 0$ for every $x \in M$, where

$$P(z) = z^n+a_1z^{n-1}+\ldots+a_n \in C^\infty(M, \boldsymbol{R})[z].$$

PROPOSITION 1. *Let f be a* (1,1) *tensor field on M satisfying the polynomial equation* (1). *Assume that there are polynomials $P_1(z), \ldots, P_r(z)$ from $C^\infty(M, \mathbf{R})$ such that*

$$P(z) = P_1(z) \ldots P_r(z)$$

and for every $x \in M$ the polynomials $P_{1_x}(z), \ldots, P_{r_x}(z)$ are mutually prime. Then, by putting

$$D_{i_x} = \ker P_{i_x}(f_x)$$

for $i = 1, \ldots, r$, $x \in M$, we obtain the almost product structure $D = (D_1, \ldots, D_r)$ on M.

Proof. By the assumption of our proposition $T_x M = \bigoplus_{i=1}^{r} D_{i_x}$ for every $x \in M$.

If $\dim D_{i_x}$ were locally constant, then it would be constant on M (M is connected) and D_i as a constant dimensional kernel of a homomorphism of the vector bundle TM would be a smooth distribution on M. Therefore it is sufficient to show that $\dim D_{i_x}$ is locally constant on M.

Let $x_0 \in M$. Denote $Q_{i_x} = P_{i_x}(f_x)$, for $i = 1, \ldots, r$, $x \in M$. First suppose that there is $1 \leqslant k \leqslant r$ such that $Q_{k_{x_0}}$ vanishes on $T_{x_0}M$. Then for every $j \neq k$ $Q_{j_{x_0}}$ is an isomorphism. Therefore Q_{j_x} are isomorphisms for every $j \neq k$ and x sufficiently close to x_0. So in this case $\dim D_{i_x}$ is constant in a neighbourhood of x_0, for every $i = 1, \ldots, r$.

Assume now that there is no $1 \leqslant k \leqslant r$ such that $Q_{k_{x_0}}$ vanishes on $T_{x_0}M$. Let $1 \leqslant i \leqslant r$. Then

$$Q_{i_x}\Big(\bigoplus_{j \neq i} D_{j_x}\Big) \subset \bigoplus_{j \neq i} D_{j_x}.$$

It follows from the fact that subspaces D_{j_x} are f_x-invariant and Q_{i_x} is a polynomial in f_x, so D_{j_x} are Q_{i_x}-invariant. Moreover, by the definition of Q_i

$$Q_{i_x} | \bigoplus_{j \neq i} D_{j_x}$$

is a monomorphism. Hence $\operatorname{im} Q_{i_x} = \bigoplus_{j \neq i} D_{j_x}$. Since $\dim \bigoplus_{j \neq i} D_{j_{x_0}} > 0$ there are vector fields $X_1, \ldots, X_m$ in a neighbourhood of x_0 such that $X_{1_{x_0}}, \ldots, X_{m_{x_0}}$ form a basis in the vector space $\bigoplus_{j \neq i} D_{j_{x_0}}$. Then

$$\{Q_{i_{x_0}}(X_{1_{x_0}}), \ldots, Q_{i_{x_0}}(X_{m_{x_0}})\}$$

is also a basis in $\bigoplus_{j \neq i} D_{j_{x_0}} = \operatorname{im} Q_{i_{x_0}}$. Hence

$$(Q_i(X_1))_{x_0} \wedge \ldots \wedge (Q_i(X_m))_{x_0} \neq 0 .$$

It follows that

$$(Q_i(X_1))_x \wedge \ldots \wedge (Q_i(X_m))_x \neq 0$$

in some neighbourhood U of x_0. Consequently $\dim \operatorname{im} Q_{i_x} \geqslant \dim \operatorname{im} Q_{i_{x_0}}$ in U. This means that $\dim D_{i_x} \leqslant \dim D_{i_{x_0}}$ for $x \in U$. Since $1 \leqslant i \leqslant r$ has been taken arbitrarily, $\dim D_{i_x} \leqslant \dim D_{i_{x_0}}$ for every $i = 1, \ldots, r$ and sufficiently near x_0. But $\bigoplus_{i=1}^{r} D_{i_x} = T_x M$, so $\dim D_{i_x}$ is constant in some neighbourhood of x_0. This finishes the proof.

PROPOSITION 2. *Let M be simply connected and let*

$$P(z) = z^n + a_1 z^{n-1} + \ldots + a_n \in C^\infty(M, \boldsymbol{C})[z].$$

If for every $x \in M$ the polynomial $P_x(z) \in \boldsymbol{C}[z]$ has n distinct roots, then $P(z)$ has n roots in $C^\infty(M, \boldsymbol{C})[z]$. If moreover the coefficients of $P(z)$ are real valued functions, then there exist functions $\xi_1, \ldots, \xi_k, \alpha_1, \beta_1, \ldots, \alpha_s, \beta_s \in C^\infty(M, \boldsymbol{R})$ such that $P(z) = (z-\xi_1) \ldots \ldots (z-\xi_k)(z^2+2\alpha_1 z+\beta_1) \ldots (z^2+2\alpha_s z+\beta_s)$, where $\alpha_j^2(x) - \beta_j(x) < 0$ for every $x \in M$ and $j = 1, \ldots, s$.

Proof. Let $x_0 \in M$ and let (V, φ) be a chart around x_0. Denote $\check{x}_0 = \varphi(x_0)$, $\check{V} = \varphi(V)$ and $\check{a}_i = a_i \circ \varphi^{-1}$. Consider the function F on $\check{V} \times \boldsymbol{C}$ defined by

$$F(x, z) = z^n + \check{a}_1(x) z^{n-1} + \ldots + \check{a}_n(x)$$

Let $\xi_1^0, \ldots, \xi_n^0$ be roots of the polynomial

$$\varkappa(z) = z^n + \check{a}_1(\check{x}_0) z^{n-1} + \ldots + \check{a}_n(\check{x}_0).$$

Clearly $F(\check{x}_0, \xi_i^0) = 0$ for $i = 1, \ldots, n$. It is easy to check that "$d_{(\check{x}_0, \xi_i^0)} F = d_{\xi_i^0} \varkappa$ is an isomorphism. In fact, the function $\varkappa$ is holomorphic, so it is sufficient to show that the complex derivative $\varkappa'(z)$ does not vanish at the point ξ_i^0. By virtue of the equality

$$\varkappa(z) = \prod_{i=1}^{n} (z - \xi_i^0),$$

we have

$$\varkappa'(z) = \prod_{j \neq i} (z - \xi_j^0) + \Big(\prod_{j \neq i} (z - \xi_j^0)\Big)'(z - \xi_i^0).$$

Therefore

$$\varkappa'(\xi_i^0) = \prod_{j \neq i} (\xi_i^0 - \xi_j^0) \neq 0,$$

because the roots $\xi_1^0, \ldots, \xi_n^0$ are mutually distinct. By the implicit function theorem it follows that for every $i = 1, \ldots, n$ there is an open neighbourhood $\check{U}_i \subset \check{V}$ of x_0 and a unique function $\check{\xi}_i$ on $\check{U}_i$ such that $\check{\xi}_i(\check{x}_0) = \xi_i^0$ and $F(x, \check{\xi}_i(x)) = 0$ in $\check{U}_i$. Let $\check{U}$ be a connected open set included in $\check{U}_1 \cap \ldots \cap \check{U}_n$ and let $U = \varphi^{-1}(\check{U})$. We define root functions $\xi_1, \ldots, \xi_n$ on U by $\xi_i = \check{\xi}_i \circ \varphi|_U$ for $i = 1, \ldots, n$.

Notice here that this consideration shows that if η_1, η_2 are root functions of $P(z)$ on M then the set

$$\{x \colon \eta_1(x) = \eta_2(x)\}$$

is open in M.

Since x_0 has been taken arbitrarily, we obtain the following fact: there is an open covering $\mathscr{U}$ of M such that for every $U \in \mathscr{U}$ there are exactly n root functions of $P(z)$ on U. Using this fact and the fact that M is simply connected it is seen that there are root functions $\xi_1, \ldots, \xi_n$ of $P(z)$ on M.

Now suppose that $a_1, \ldots, a_n$ are real valued functions. If ξ is a root function, then the conjugate function $\bar{\xi}$ is also a root function. It has already been remarked, that the set

$$\{x \colon \xi(x) = \bar{\xi}(x)\}$$

is open in M. But it is also closed in M, so by the connectedness of M, if $\xi(x_0) \in \mathbf{R}$ at some point $x_0 \in M$, then $\xi(x) \in \mathbf{R}$ for every $x \in M$, and consequently if $\xi(x_0) \in \mathbf{C} \setminus \mathbf{R}$ for some $x_0 \in M$, then $\xi(x) \in \mathbf{C} \setminus \mathbf{R}$ for every $x \in M$. Hence we may number root functions $\xi_1, \ldots, \xi_n$ is such a way that $\xi_1, \ldots, \xi_k$ have real values on M and

$$\xi_{k+2} = \bar{\xi}_{k+1}, \ldots, \xi_n = \bar{\xi}_{n-1} .$$

Putting

$$\alpha_i = -\tfrac{1}{2}(\xi_{k+i} + \xi_{k+i+1}) \quad \text{and} \quad \beta_i = \xi_{k+i} \cdot \xi_{k+i+1}$$

for $i = 1, \ldots, s = n-k-1$, we obtain functions satisfying the required condition. This completes the proof.

Let f be a (1,1) tensor field on M satisfying the polynomial equation (1). Assume that for every $x \in M$ the polynomial $P_x(z)$ has only simple roots. Let $x \in M$ and let

$$\varrho_{1x}, \ldots, \varrho_{kx}, \qquad \gamma_{1x}, \delta_{1x}, \ldots, \gamma_{sx}, \delta_{sx}$$

be such real number that

$$P_x(z) = (z - \varrho_{1x}) \ldots (2 - \varrho_{kx})(z^2 + 2\gamma_{1x} z + \delta_{1x}) \ldots (z^2 + 2\gamma_{sx} z + \delta_{sx}) .$$

Then $T_x M = \bigoplus_{i=1}^{n} T_{i_x}$, where $T_{i_x} = \ker R_{i_x}(f_x)$ and

$$R_{i_x}(z) = \begin{cases} z - \varrho_{i_x}, & 1 \leqslant i \leqslant k, \\ z^2 + 2\gamma_{i-k} z + \delta_{i-k}, & k < i \leqslant n . \end{cases}$$

Denoting by $Q_{1_x}, \ldots, Q_{n_x}$ the projectors of the decomposition $T_x M = \bigoplus_{i=1}^{n} T_{i_x}$, we define

$$J_x = \sum_{i=1}^{s} \frac{f_x + \gamma_{ix} I_x}{\sqrt{\delta_{ix}^2 - \gamma_{ix}^2}} Q_{(i+k)_x}, \tag{2}$$

where I_x is the identity mapping of $T_x M$. It is easy to verify that $J_x^3 + J_x = 0$.

Since the polynomial $P_x(z)$ has only simple roots for every $x \in M$, $P(z)$ is locally decomposable by Proposition 2., i.e. for every $x \in M$ there is an open neighbourhood U of x and functions $\xi_1, \ldots, \xi_k, \alpha_1, \beta_1, \ldots, \alpha_s, \beta_s \in C^\infty(U, \mathbf{R})$ such that

$$P(z)|_U = (z - \xi_1) \ldots (z - \xi_k)(z^2 + 2\alpha_1 z + \beta_1) \ldots (z^2 + 2\alpha_s z + \beta_s) .$$

Such a decomposition defines on U the almost product structure $D = (D_1, \ldots, D_s)$, as in Proposition 1.,

Notice that if M is simply connected, then f defines the almost product structure on the whole M.

Going back to our consideration, we denote by $P_1, \ldots, P_n$ projectors of the almost product structure D. These projectors are smooth (1,1) tensor fields on U. It is obvious that

$$\sum_{i=1}^{s} \frac{f_x + \gamma_{i_x} I_x}{\sqrt{\delta_{i_x}^2 - \gamma_{i_x}^2}} Q_{i+k_x} = \sum_{i=1}^{s} \frac{f_x + \alpha_i(x) I_x}{\sqrt{\beta_i^2(x) - \alpha_i^2(x)}} P_{i+k_x},$$

for every $x \in M$. Consequently J defined as

$$J\colon M \ni x \to J_x ,$$

where J_x is given by the formula (2) is a smooth (1,1) tensor field on M. J is an almost complex structure if only there is $x \in M$ such that the polynomial $P_x(z)$ has no real root.

Reference

[1] J. Vanžura, *Integrability conditions for polynomial structures*, Kodai Math. Sem. Rep. 27 (1976) 42-50.

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