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On Fornaess' Imbedding Theorem

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Abstract. In [4] Fornaess proved that for every strictly pseudoconvex domain D in C^n there exists an embedding of D in some strictly convex domain C in higher-dimensional space C^m. He proved also the existence of the continuous and linear extension operator $L: H^\infty(D) \to H^\infty(C)$ such that $L(A(D)) \subset A(C)$. In this paper we prove the same property with respect to the spaces $H^{\infty,k}(D)$, $A^k(D)$ and $\Lambda_t(D)$, under some regularity assumptions on ∂D. As an application we give an alternative proof of the decomposition property for the spaces $H^{\infty,k}(D)$, $A^k(D)$ and $\Lambda_t(D)$ [2], together with some additional properties of decomposition operators.

1. Notations and definitions. For every point $z \in C^n$ and every $r>0$, $B(z,r)$ denotes the euclidean ball with radius r centered at z.

If $\alpha, \beta \in Z^n_+$ are multiindices, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, and j is an integer, $1 \leqslant j \leqslant n$, we set $\alpha+\beta = (\alpha_1+\beta_1, \ldots, \alpha_n+\beta_n)$, $e_j = (0, \ldots, 1, \ldots, 0)$ (1 on the j-th place), $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $\alpha-\beta = (\alpha_1-\beta_1, \ldots, \alpha_n-B_n)$ if $\alpha_j \geqslant \beta_j$ for each $j = 1, \ldots, n$.

We use the standard abbreviations: For any $\alpha, \beta \in Z^n_+$, $1 \leqslant j \leqslant n$, let $D^j = \frac{\partial}{\partial z_j}$, $\bar{D}^j = \frac{\partial}{\partial \bar{z}_j}$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$, etc. Moreover, we set the subscript in order to indicate with respect to which group of variables the differentiation holds, e.g. D^j_z, $\bar{D}^\beta_\xi$.

If X is a set and f is a complex-valued function defined on X, let

$$||f||_X = \sup_X |f|.$$

Let D be an open set in C^n. For $f \in \mathscr{C}^k(D)$, $k = 1, 2, \ldots$, set

$$||f||_{D,k} = \sum_{|\alpha|+|\beta| \leqslant k} \sup_D |D^\alpha \bar{D}^\beta f|.$$

If $f(\xi, z)$ is of class $\mathscr{C}^k$ with respect to $(\xi, z) \in D \times D$ and of class $\mathscr{C}^l$ in $z \in D$, we define

$$||f||_{D,k_\xi,l_z} = \sum \sup_{D \times D} |D^{\alpha'}_\xi \bar{D}^{\alpha''}_\xi D^{\beta'}_z \bar{D}^{\beta''}_z f|,$$

where the summation is extended over all multiindices α', α'', β', β'' such that $|\alpha'|+|\alpha''|+ +|\beta'|+|\beta''| \leqslant l$ and $|\alpha'|+|\alpha''| \leqslant k$.

If t is a positive real number, which is not an integer, and k is a non-negative integer with $k<t<k+1$, we denote by $\tilde{\Lambda}_t(D)$ the space of all functions $f \in \mathscr{C}^k(\bar{D})$ such that for any $\alpha, \beta \in \mathbf{Z}_+^n$ with $|\alpha|+|\beta| = k$, there exists a constant $c_{\alpha\beta}>0$ such that

$$|D^\alpha \bar{D}^\beta f(z) - D^\alpha \bar{D}^\beta f(z')| \leqslant c_{\alpha\beta}|z-z'|^{t-k}.$$

$\tilde{\Lambda}_t(D)$ is a Banach space with the norm

$$||f||_{D,t} = ||f||_{D,k} + \sum_{|\alpha|+|\beta|=k} \sup_{z,z' \in D} |D^\alpha \bar{D}^\beta f(z) - D^\alpha \bar{D}^\beta f(z')|/|z-z'|^{t-k}.$$

Let t be as above, and let l be a non-negative integer. We define $\tilde{\Lambda}_{t,l}(D)$ to be the space of all functions $f(\xi, z)$ defined in $D \times D$, such that

$$||f||_{D,t,l} = \sum_{|\alpha|+|\alpha'| \leqslant k} ||D_\xi^\alpha \bar{D}_\xi^{\alpha'} f||_{D\times D} + \sum_{\substack{|\beta|+|\beta'| \leqslant l \\ |\alpha|+|\alpha'|=k}} \sup_{z \in D} ||D_\xi^\alpha \bar{D}_\xi^{\alpha'} D_z^\beta \bar{D}_z^{\beta'} f(\cdot, z)||_{D,t} < \infty$$

(see [2], p. 546). $\tilde{\Lambda}_{t,l}(D)$ is a Banach space with the norm $|| \; ||_{D,t,l}$.

By $\mathcal{O}(D)$ we shall denote the space of all functions holomorphic in D.

For every $k = 0, 1, 2, \ldots$, set $A^k(D) = \mathcal{O}(D) \cap \mathscr{C}^k(\bar{D})$, and let $H^{\infty,k}(D)$ be the algebra of all functions holomorphic in D such that their derivatives of order $\leqslant k-1$ extend continuously to $\bar{D}$, and the derivatives of order k are bounded in D. If D is a bounded domain in $\mathbf{C}^n$, then $A^k(D)$ and $H^{\infty,k}(D)$ are Banach algebras under the norm $|| \; ||_{D,k}$. We write $A(D) = A^0(D)$ and $H^\infty(D) = H^{\infty,0}(D)$.

Similarly, for $t>0$, let $\Lambda_t(D) = \mathcal{O}(D) \cap \tilde{\Lambda}_t(D)$. $\tilde{\Lambda}_t(D)$ is a Banach space with the norm $|| \; ||_{D,t}$.

Let U and V be two sets in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. By a differential form of order $((p,q)_\xi, (0,0)_z)$ in $U \times V$ with coefficients in some space F of functions defined in $U \times V$ (which will be specified in every case) we mean the form

$$\omega(\xi, z) = \Sigma' \omega_{IJ}(\xi, z) d\xi^I \wedge d\bar{\xi}^J$$

where $(\xi, z) \in U \times V$, the coefficients $\omega_{IJ}(\xi, z) \in F$ and Σ' denotes that the summation extends over all sequences $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ with $1 \leqslant i_1 < \ldots < i_p \leqslant n$ and $1 \leqslant j_1 < \ldots < j_q \leqslant n$, and $d\xi^I = d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p}$, $d\bar{\xi}^J = d\bar{\xi}_{j_1} \wedge \ldots \wedge d\bar{\xi}_{j_q}$.

We assume that the differential operators, as applied to differential forms, act on their coefficients.

A bounded domain $D \subset \mathbf{C}^n$ is called a domain with $\mathscr{C}^k$ boundary if there exists a neighborhood U of ∂D and a real-valued function $\varrho \in \mathscr{C}^k(U)$ such that

(i) $D = (D \setminus U) \cup \{z \in U\colon \varrho(z) < 0\}$,

(ii) grad $\varrho(z) \neq 0$ for $z \in \partial D$.

The function which satisfies the above conditions is called a defining function for D. Moreover, for ε sufficiently close to 0, we put

$$D_\varepsilon = (D \setminus U) \cup \{z \in U\colon \varrho(z) < \varepsilon\}.$$

Let D be a domain in $\mathbf{C}^n$ with $\mathscr{C}^k$ boundary, $k \geqslant 2$. D is called strictly pseudoconvex (with $\mathscr{C}^k$ boundary) if there exists a defining function for D of class $\mathscr{C}^k$, which is strictly plurisubharmonic in a neighborhood of ∂D. If the defining function ϱ for D can be chosen in such a way, that $\varrho \in \mathscr{C}^k(\mathbf{C}^n)$ satisfies the condition $\lim \varrho(z) = +\infty$ and has a real hessian positive definite at every point of $\mathbf{C}^n$, and is such that $D = \{z \in \mathbf{C}^n: \varrho(z) < 0\}$, then D is called a strictly convex domain with $\mathscr{C}^k$ boundary ([4], p, 529).

If M is a real $\mathscr{C}^1$ submanifold of $\mathbf{C}^n$ and $x \in M$, we denote by $T_x(M)$ a tangent space to M at x. $T_x^{\mathbf{C}}(M)$ will denote the maximal complex subspace of $T_x(M)$.

We use the following abbreviations: If $\xi, z \in \mathbf{C}^n$ and $\eta \in \mathbf{Z}_+^n$, we write

$$(\xi - z)^\eta = (\xi_1 - z_1)^{\eta_1} \dots (\xi_n - z_n)^{\eta_n}, \ (\bar{\xi} - \bar{z})^\eta = (\bar{\xi}_1 - \bar{z}_1)^{\eta_1} \dots (\bar{\xi}_n - \bar{z}_n)^{\eta_n},$$

and for every j, $1 \leqslant j \leqslant n$,

$$d\xi[j] = d\xi_1 \wedge \dots \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \dots \wedge d\xi_n,$$

and similarly

$$d\bar{\xi}[j] = d\bar{\xi}_1 \wedge \dots \wedge d\bar{\xi}_{j-1} \wedge d\bar{\xi}_{j+1} \wedge \dots \wedge d\bar{\xi}_n;$$

moreover, we write

$$d\xi = d\xi_1 \wedge \dots \wedge d\xi_n \quad \text{and} \quad d\bar{\xi} = d\bar{\xi}_1 \wedge \dots \wedge d\bar{\xi}_n.$$

If F and G are non-negative functions defined in K, then $F(x) \lesssim G(x)$ means that there exists a constant $c > 0$ such that $F(x) \leqslant c \cdot G(x)$ for every $x \in K$.

If V and U are open subsets of $\mathbf{C}^n$, we write $V \subset\subset U$ to denote that V is a relatively compact subset of U.

2. Introduction. Fornaess proved in [4] the following theorem on the embedding of strictly pseudoconvex domain in $\mathbf{C}^n$ in strictly convex domains:

THEOREM A [4, Thm 9]. *Let $D \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with $\mathscr{C}^k$ boundary, $k = 2, 3, \dots, \infty$. Then there exist: a Stein neighborhood $\tilde{D}$ of $\bar{D}$, an integer m with $n \leqslant m$, a holomorphic mapping $\psi: \tilde{D} \to \psi(\tilde{D}) \subset \mathbf{C}^m$ which maps $\tilde{D}$ biholomorphically onto a closed submanifold of $\mathbf{C}^m$, and a strictly convex domain $C \subset \mathbf{C}^m$ with $\mathscr{C}^k$ boundary such that $\psi(D) \subset C$, $\psi(\tilde{D} \setminus \bar{D}) \subset \mathbf{C}^m \setminus \bar{C}$ and $\psi(\tilde{D})$ intersects ∂C transversally at each point $p \in \partial C \cap \psi(\tilde{D})$* (i.e. *for each $p \in \partial C \cap \psi(\tilde{D})$, $T_p(\partial C) + T_p(\psi(\tilde{D})) = \mathbf{C}^m$*).

Fornaess proved also that in the above situation there exists a linear and continuous extension operator $L: H^\infty(\psi(D)) \to H^\infty(C)$ such that $L(A(\psi(D))) \subset A(C)$ ([4, Thm 4], p. 563).

In this paper we show that under certain assumptions on the smoothness of ∂D, the operator L has additional regularity properties:

THEOREM 1. *Let $D \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with $\mathscr{C}^p$ boundary. Let $\tilde{D}$, m, ψ and C be as in the assertion of Theorem* A.

(a) *If $p \geqslant k+5$, then $L(H^{\infty,k}(\psi(D))) \subset H^{\infty,k}(C)$ and $L: H^{\infty,k}(\psi(D)) \to H^{\infty,k}(C)$ is linear and continuous extension operator. Moreover, $L(A^k(\psi(D))) \subset A^k(C)$.*

(b) *If* $p \geq k+6$, *and* $t \in R$ *with* $k<t<k+1$, *then* $L(\Lambda_t(\psi(D))) \subset \Lambda_t(C)$, *and* $L\colon \Lambda_t(\psi(D)) \to \Lambda_t(C)$ *is a linear and continuous extension operator.*
(We set here $H^{\infty,k}(\psi(D)) = \{f \circ \psi^{-1}\colon f \in H^{\infty,k}(D)\}$, and $A^k(\psi(D))$ and $\Lambda_t(\psi(D))$ are defined similarly).

As an application we give the alternative proof of the following result on the decomposition properties of some spaces of holomorphic functions in strictly pseudoconvex domains:

THEOREM 2 [2, Thm 4] (see also [10]). *Let* $D \subset C^n$ *be a strictly pseudoconvex domain with* $\mathscr{C}^p$ *boundary*

(a) *If* $p \geq k+5$, *then for each* $f \in A^k(D)$ *there exist functions* $f_i(z, s)$, $i = 1, \ldots, n$, *holomorphic in* $D \times D$ *and of class* $\mathscr{C}^k$ *in the set* $Q = (\bar{D} \times \bar{D}) \setminus \{(x, x)\colon x \in \partial D\}$, *such that*

$$f(z) = f(s) + \sum_{i=1}^{n} (z_i - s_i) \cdot f_i(z, s) \tag{2.1}$$

for every $(z, s) \in Q$. *Moreover, for every fixed* $s \in D$ *the mapping*

$$R_k(s)\colon A^k(D) \ni f \to (f_1(\cdot, s), \ldots, f_n(\cdot, s)) \in (A^k(D))^n$$

is linear and continuous.

(b) *Let* p *be as above. Then for every* $f \in H^{\infty,k}(D)$ *there exist functions* $f_i(z, s)$, $i = 1, \ldots, n$, *holomorphic in* $D \times D$, *such that for every fixed* $s \in D$ (*resp.* $z \in D$), $f_i(\cdot, s) \in H^{\infty,k}(D)$ ($f_i(z, \cdot) \in H^{\infty,k}(D)$), *and which satisfy* (2.1) *for every* $(z, s) \in D \times D$. *Moreover, given* $s \in D$, *the mapping*

$$Q_k(s)\colon H^{\infty,k}(D) \ni f \to (f_1(\cdot, s), \ldots, f_n(\cdot, s)) \in (H^{\infty,k}(D))^n$$

is linear and continuous.

(c) *If* $p \geq k+6$ *and* $k<t<k+1$, *then for every* $f \in \Lambda_t(D)$ *one can find* $f_i(z, s) \in \mathcal{O}(D \times D)$ *which satisfy* (2.1) *in* $D \times D$, *such that for each fixed* s, $z \in D$, $f_i(\cdot, s)$ *and* $f_i(z, \cdot)$ *are in* $\Lambda_t(D)$, *and the corresponding mapping*

$$S(s)\colon \Lambda_t(D) \ni f \to (f_1(\cdot, s), \ldots, f_n(\cdot, s)) \in (\Lambda_t(D))^n$$

is linear and continuous.

The above theorem was proved by Ahern and Schneider in [2], under the assumption that ∂D is of class $\mathscr{C}^{k+3}$. Therefore, our result is weaker than that of [2], and the proof is more complicated. However, we obtain more informations about the functions $f_i(z, s)$. It is conjectured in [2] that the functions $f_i(z, s)$, obtained by methods of [2], are in $A^{k-1}(D \times D)$ (resp. in $H^{\infty,k-1}(D \times D)$ or in $\Lambda_{t-1}(D \times D)$) if $f \in A^k(D)$, $H^{\infty,k}(D)$ or $\Lambda_t(D)$ respectively. The proof of Theorem 2 presented in this paper yields the functions $f_i(z, s)$ which satisfy the conjectured properties. Moreover, we obtain some estimates on the norm of the linear operators $R_k(s)$, $Q_k(s)$ and $S_t(s)$:

PROPOSITION 3. *If D is a strictly pseudoconvex domain in C^n such that ∂D satisfies the regularity properties of Theorem 2, then there exist constants M_1, M_2 and M_3, which depend only on D, such that for each $s \in D$,*

$$R_k(s) \leqslant \frac{M_1}{d_D(s)^2}, \quad Q_k(s) \leqslant \frac{M_2}{d_D(s)^2}, \quad \text{and} \quad S_t(s) \leqslant \frac{M_3}{d_D(s)^2}.$$

(Here $d_D(s)$ denotes the Euclidean distance from s to ∂D).

3. Proof of Theorem 1. Let $D \subset C^n$ be a strictly pseudoconvex domain with $\mathscr{C}^N$ boundary. Choose $\tilde{D}$, m, ψ and C according to Theorem A. For convenience, we recall the construction of the linear and continuous extension operator L: $H^\infty(\psi(D)) \to H^\infty(C)$, given in [4], p. 562.

Let $\sigma \in \mathscr{C}^N(C^m)$ be a strictly convex function which is defining for a domain C. In [4], pp. 563–564 the following properties of D, ψ and C are proved:

(A) For each $\zeta_0 \in \partial D$ there exist a neighborhood Ω of ζ_0 in $\tilde{D}$ and functions a_i: $\Omega \to C^m$, $a_i \in \mathscr{C}^{N-1}(\Omega)$, $i = 1, \ldots, m$ such that for each $\zeta \in \Omega$:

(i) the vectors $a_1(\zeta), \ldots, a_m(\zeta)$ span C^m,

(ii) $\{a_1(\zeta), \ldots, a_{m-1}(\zeta)\}$ is an orthonormal basis for the space

$$\{(\eta_1, \ldots, \eta_m) \in C^m\colon \sum_{i=1}^{m} (D^i\sigma)(\psi(\zeta))\eta_i = 0\},$$

(iii) $\{a_{m-n+1}(\zeta), \ldots, a_m(\zeta)\}$ is an orthonormal basis for $T^{C}_{\psi(\zeta)}(\psi(\tilde{D}))$.

(B) There exists a neighborhood V of $\xi_0 = \psi(\zeta_0)$ such that for each $z \in V$ there exists exactly one point $\xi_z = \psi(\zeta_z) \in \psi(\tilde{D})$ such that

$$z \in \xi_z + \operatorname{span}\{a_1(\zeta_z), \ldots, a_{m-n}(\zeta_z)\}. \tag{3.1}$$

The mappings

$$z \to \zeta_z, \quad z \to \xi_z \tag{3.2}$$

are $\mathscr{C}^{N-1}$ in V. Moreover, if $z \in V \cap (\bar{C} \setminus \psi(\partial D))$, then $\xi_z \in \psi(D)$.

Let $f(\xi, z) = \sum_{i=1}^{m} (D^i\sigma)(\xi)(\xi_i - z_i)$. It is proved in [4] that the function

$$\Phi(\zeta, y) = 2f(\psi(\zeta), \psi(y)) \in \mathscr{C}^{k-1}(\tilde{D} \times \tilde{D})$$

satisfies the following properties:

LEMMA B [5], [4]. (i) *$\Phi(\zeta, \zeta) = 0$ for every $\zeta \in \tilde{D}$.*

(ii) *For each compact subset $K \subset D$ there exists a constant $\gamma = \gamma(K) > 0$ such that*

$$\operatorname{Re}\Phi(\zeta, y) \geqslant \varrho(\zeta) - \varrho(y) + \gamma|\zeta - y|^2, \quad (\zeta, y) \in K \times K,$$

where $\varrho = \sigma \circ \psi$.

(iii) *There exists a neighborhood U of ∂D such that the vectors*

$$\operatorname{grad} \varrho(\zeta) \quad \text{and} \quad \operatorname{grad}_\zeta \operatorname{Im} \Phi(\zeta, y)|_{\zeta=y}$$

are linearly independent (over $\boldsymbol{R}$) *at every point* $\zeta \in U$.

(iv) *There exist functions* $F_{ij} \in \mathcal{O}(\tilde{D} \times \tilde{D})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ *such that*

$$\Phi(\zeta, y) = \sum_{j=1}^{n} (\zeta_j - y_j) P_j(\zeta, y), \quad (\zeta, y) \in \tilde{D} \times \tilde{D},$$

with

$$P_j(\zeta, y) = \sum_{i=1}^{m} (D^i \sigma)(\psi(\zeta)) F_{ij}(\zeta, y).$$

(v) *For each* $\zeta \in \partial D$ *there exists* j, $1 \leqslant j \leqslant n$, *such that*

$$D_y^j \Phi(\zeta, y)|_{\zeta=y} \neq 0.$$

This lemma says in particular that $\Phi(\zeta, y)$ is an integral kernel for a domain D, introduced by Henkin in [5].

It follows from [15, Lemma I.1] that for every $h \in H^\infty(D)$,

$$h(y) = \int_{\partial D} h(\zeta) C(\zeta, y), \quad y \in D, \tag{3.3}$$

where $h(\zeta) \in L^\infty(\partial D)$ denotes the boundary values of h on ∂D, which exist for almost all $\zeta \in \partial D$ by [14], and

$$C(\zeta, y) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{(-1)^{j-1} P_j(\zeta, y) \wedge \bigwedge_{k \neq j} \bar{\partial}_\zeta P_k(\zeta, y) \wedge d\zeta}{\Phi(\zeta, y)^n}. \tag{3.4}$$

Let $\tilde{F}_{ij}$, $\tilde{\zeta}_j$ be holomorphic functions in $\boldsymbol{C}^m \times \boldsymbol{C}^m$ and in $\boldsymbol{C}^m$ respectively, such that $\tilde{F}_{ij}(\psi(\zeta), \psi(y)) = F_{ij}(\zeta, y)$ for $(\zeta, y) \in \tilde{D} \times \tilde{D}$ and $\tilde{\zeta}_j(\psi(\zeta)) = \zeta_j$ for $\zeta \in \tilde{D}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. For $(\xi, z) \in \boldsymbol{C}^m \times \boldsymbol{C}^m$, put

$$\tilde{g}_j(\xi, z) = \sum_{i=1}^{m} (D^i \sigma)(\xi) \tilde{F}_{ij}(\xi, z), \quad j = 1, \ldots, n.$$

Define the kernel $K(\xi, z)$ by

$$K(\xi, z) = \frac{\sum_{j=1}^{n} (-1)^{j-1} \tilde{g}_j(\xi, z) \wedge \bigwedge_{k \neq j} \bar{\partial}_\xi \tilde{g}_k(\xi, z) \wedge d\tilde{\zeta}_1(\xi) \wedge \ldots \wedge d\tilde{\zeta}_n(\xi)}{(2f(\xi, z))^n},$$

for $(\xi, z) \notin f^{-1}(0)$. Let $h \in H^\infty(\psi(D))$. It follows from (3.3) and from the properties of $K(\xi, z)$ that the function

$$Lh(z) = c_n \int_{\psi(\partial D)} h(\xi) K(\xi, z), \quad z \in C,$$

is holomorphic in C and $Lh(z) = h(z)$ for $z \in \psi(D)$ (we put $c_n = \dfrac{(n-1)!}{(2\pi i)^n}$ and

$$h(\xi) \in L^\infty(\psi(\partial D))$$

denotes the boundary values of $h \circ \psi$ on ∂D, mapped onto $\psi(\partial D)$ by ψ).

THEOREM C ([4, Thm 4], p. 563). *L is a linear and continuous extension operator L: $H^\infty(\psi(D)) \to H^\infty(C)$. Moreover, $L(A(\psi(D))) \subset A(C)$.*

Therefore, Theorem 1 states the additional regularity properties of L, provided that ∂D is smooth enough.

The proof of Theorem 1 is based on Fornaess' construction used in the proof of Theorem C, and on the techniques developed by Siu in [13]. In the sequel of this paragraph, we assume that D is a strictly pseudoconvex domain in $\mathbf{C}^n$, and that $\tilde{D}$, m, ψ and C are chosen according to Theorem A. We need first some lemmas:

LEMMA 4. *Let ∂D be of class $\mathscr{C}^{k+4}$, $k = 1, 2, \ldots$, and let $f \in H^{\infty,k}(D)$. Suppose that $g(\zeta, y) \in \mathscr{C}^{k+1}(\bar{D} \times \bar{D})$ and is of class $\mathscr{C}^{k+2}$ with respect to y. Define the function f_g by*

$$f_g(y) = \int_{\partial D} f(\zeta) g(\zeta, y) C(\zeta, y), \quad y \in D,$$

where $C(\zeta, y)$ is given in (3.4).

Then f_g extends to the function of class $\mathscr{C}^{k-1}(\bar{D})$ and has the derivatives of order k bounded in D, and

$$\|f_g\|_{D,k} \leqslant c \|f\|_{D,k} \|g\|_{D,(k+1)_\zeta,(k+1)} \tag{3.5}$$

for some $c > 0$ independent of f and g.

This lemma is in [2]. It is stated there for domains with $\mathscr{C}^\infty$ boundary and for $g(\zeta, y) \in \mathscr{C}^\infty(\bar{D} \times \bar{D})$, but the regularity assumption given here is enough to prove the result (see also [10]).

LEMMA 5. *Suppose that ∂D is of class $\mathscr{C}^N$, and let l, p be non-negative integers such that $l \leqslant 2(p-n)+1$ and $2(p-n)+4-l \leqslant N$. Fix $\xi_0 \in \psi(\partial D)$ and let $A(\zeta, z)$ be a differential form of order $((n, n-1)_\zeta, (0, 0)_z)$ with coefficients in $\mathscr{C}^{2(p-n)-l+2}(D_1 \times W)$, where D_1 is a neighborhood of $\bar{D}$ and W is a neighborhood of ξ_0. Let $g(\zeta, z) \in \mathscr{C}^1(D_1 \times W)$ and holomorphic in ζ satisfy the condition $|g(\zeta, z)| \leqslant c|\psi(\zeta) - z|^l$ for $(\zeta, z) \in D_1 \times W$.*

Then there exists a neighborhood U of ξ_0 in $\mathbf{C}^m$ such that the function

$$F(z) = \int_{\partial D} \frac{g(\zeta, z) A(\zeta, z)}{f(\psi(\zeta), z)^p}, \quad z \in V \cap C,$$

extends continuously to $V \cap \bar{C}$.

The proof is a convenient modification of the proof of [13, Prop. 3.3].

LEMMA 6. *Suppose that ∂D is of class $\mathscr{C}^2$. Fix $\zeta_0 \in \partial D$ and let V be a neighborhood of $\xi_0 = \psi(\zeta_0)$ which satisfies the condition* (B). *Let $M(\xi, z)$ be a product of r factors of type $|\xi-\zeta_z|$ or $|\xi-z|$, $\xi \in \psi(\partial D)$, $z \in (V \cap \bar{C}) \setminus \psi(\partial D)$. ($\xi_z$ is defined in* (3.1)).

Then there exists a neighborhood W of ξ_0 such that for every $z \in (W \cap \bar{C}) \setminus \psi(\partial D)$,

$$|\xi_z - z| \int_{\partial D} \frac{M(\psi(\zeta), z)}{|f(\psi(\zeta), z)|^{n+p}} d\sigma(\zeta) \leqslant c$$

and

$$|\xi_z - z| \int_{\partial D} \frac{|\psi(\zeta) - \xi_z| \, M(\psi(\zeta), z)}{|f(\psi(\zeta), z)|^{n+p+1}} d\sigma(\zeta) \leqslant c$$

for some constant $c>0$ independent of ξ_0 and z. (Here $d\sigma(\zeta)$ denotes the volume form on ∂D).

This lemma is implicitly contained in [4], pp. 565–566.

Now we show that the estimates similar to (3.5) are valid for points on $\psi(D)$:

LEMMA 7. *Suppose that ∂D is of class $\mathscr{C}^{k+4}$, $k = 0, 1, \ldots$ Then there exists a constant $c>0$ such that for each $\alpha \in Z_+^m$, $|\alpha| \leqslant k$, and for each $h \in H^{\infty,k}(\psi(D))$,*

$$\|D^\alpha Lh\|_{\psi(D)} \leqslant c \|h \circ \psi\|_{D,k}.$$

Proof. Since

$$D^\alpha Lh(z) = \sum_{p=0}^{|\alpha|} \int_{\psi(\partial D)} h(\xi) \frac{A_p(\xi, z)}{f(\xi, z)^{n+p}}$$

where each $A_p(\xi, z)$ is a differential form of order $((n, n-1)_\xi, (0,0)_z)$ with coefficients of class $\mathscr{C}^{k+2}(\boldsymbol{C}^m \times \boldsymbol{C}^m)$ and holomorphic with respect to z, it is sufficient to prove that for every $k = 0, 1, \ldots$, for every $g \in H^{\infty,k}(D)$ and for every differential form $M(\zeta, y)$ of order $((n, n-1)_\zeta, (0,0)_y)$ with coefficients of class $\mathscr{C}^{k+2}(\tilde{D} \times \tilde{D})$ and $\mathscr{C}^\infty$ in y, the function

$$F(y) = \int_{\partial D} g(\zeta) \frac{M(\zeta, y)}{\Phi(\zeta, y)^{n+k}}, \qquad y \in D$$

is bounded in D and

$$\|F\|_D \leqslant c \|g\|_{D,k}$$

for some c independent of g. We shall show that for any $\zeta_0 \in \partial D$ there exists a neighborhood V of ζ_0 and a constant $c>0$ such that

(3.6) $$|F(y)| \leqslant c \|g\|_{D,k}$$